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WHAT IS AN ANGLE?*

HANS ZASSENHAUS, McGill University

1. Introduction. In this paper I plan to present to you my ideas on how the measurement of angles may be suitably introduced in our time. I am, however, fully aware that some of you rather desire an "angle" from which to look at what is taking place in modern algebra. I ask you, therefore, to accept with patience my diversions from the geometrical angle to the philosophical one and back to the subject matter of this paper.

As possible concepts on which the definition of an angle may be based there come to mind:

1) A pair of intersecting straight lines,
2) A pair of rays from the same origin,
3) An open convex part of the plane bounded by a pair of rays with the same origin, called *angular space*,
4) A rotation,
5) A circular arc,
6) An isosceles triangle.

Any one of these six concepts may serve to develop the notion of angles in the plane and, in fact, has played its role at one time or another. However, before accepting what other times have found suitable we want to understand the situation thoroughly. After consulting the books we come together and try to develop a picture which can be perceived and assimilated by *our* minds. This picture will, quite likely, bear features of every one of the six concepts mentioned above.

Due to the fact that so many concepts of what an angle might be are currently circulating, we feel an urge to abandon the visual concepts altogether and occupy ourselves merely with the essential properties of angles which actually do not depend on the special geometrical figures from which the angles originate. The response to this urge consists of the *axiomatic approach*.

We take it for granted that there is such a thing as an angle and we try to expose in simple terms its fundamental properties from which all other properties will follow as logical consequences.

What is simple? Is it a good quality or a bad one? The answer to the second question depends very much on the answer to the first. I will not pretend that there is either merit or reward in being a simpleton, although it seems to be a healthy state. However, a few ideas, well connected and forcefully expounded, do a lot more than a bagful of tricks. This, in my opinion, is a psychological law. We rejoice in repetitions carried on to higher levels, but we are unwilling to pursue new ways of thinking where there is no necessity. I think that simplicity in the approach to the solutions of our problems is a great human virtue and no sacrifice should be too big to attain it.

* This paper was originally an address delivered to the Students' Mathematical Society at McGill University.

2. Axioms. If we think of an angle as associated with a rotation we come to consider *oriented angles*. Also, the existence of an addition of angles satisfying all the laws of an additive group is suggested. On the other hand the concept of an angle as being associated with an angular space suggests the idea that angles can be ordered according to their size. The property of order and the group property seem to contradict each other since the repeated addition of positive angles would lead to negative ones if addition could be carried out unrestrictedly. Hence, we have to be careful in the handling of the addition. Mustering our experiences with angles as they present themselves in ordinary geometrical experience we arrive at the following axioms:

(A) There are certain geometrical things called *angles* among which there is given an equality relation satisfying the three rules of reflexivity, symmetry and transitivity. There are at least two different angles. We denote angles by small Greek letters.

(B) For certain ordered pairs of angles, α, β, a third angle called the sum of α and β is defined and may be denoted by $\alpha+\beta$ so that:

(1) $\alpha+\beta=\alpha'+\beta'$ whenever $\alpha=\alpha'$, $\beta=\beta'$ and $\alpha+\beta$ or $\alpha'+\beta'$ is defined.

(2) If $\alpha+\beta$ and $\beta+\gamma$ are defined and if at least one of $(\alpha+\beta)+\gamma$, $\alpha+(\beta+\gamma)$ is defined, then both are defined and they are equal, *i.e.*, $(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)$.

(3) There is a zero angle, 0, satisfying $\alpha+0=\alpha$ for all angles α.

(4) For every angle α there is an opposite angle $-\alpha$ satisfying

$$\alpha+(-\alpha)=0.$$

(C) Every angle, α, has a signature, sgn α, which is either 1 (positive angles), 0 (zero angles), or -1 (negative angles) so that:

(1) sgn $(\alpha+\beta)=$sgn $\alpha=$sgn β if sgn $\alpha=$sgn β and $\alpha+\beta$ is defined.

(2) $\alpha+\beta$ is defined if sgn $\alpha\neq$sgn β.

(3) $\beta+\alpha$ is defined if $\alpha+\beta$ is defined and sgn $(\beta+\alpha)=$sgn $(\alpha+\beta)$.

(4) Every non-empty system of positive angles has a least upper bound.

Regarding (C4) we define: $\alpha\geqq\beta$ if either $\alpha=\beta$, or $1=$sgn $\alpha\neq$sgn β, or $\alpha+(-\beta)>0$.

For a given non-empty set, S, of angles, an upper bound is defined as an angle λ for which $\lambda\geqq\alpha$ for all $\alpha\in S$. An upper bound, λ, of S is called the least upper bound of S (l.u.b. S) if for any upper bound λ' of S, $\lambda'\geqq\lambda$.

3. Model. Before we study the implications of the axioms A, B and C let us outline the analytical model to which they are related.

The points, P, of the Cartesian plane are given as ordered pairs of real numbers x_1, x_2 called the coördinates of P: $P=(x_1, x_2)$.

The distance between two points $P=(x_1, x_2)$ and $Q=(y_1, y_2)$ is defined as the non-negative real number

$$\overline{PQ} = \left| \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} \right|.$$

The rigid movements of the Cartesian plane are defined as distance-preserving maps: $P \to \sigma P$ of the set of all points into itself so that $\overline{\sigma P \sigma Q} = \overline{PQ}$.

It follows that the coördinates x_1', x_2' of the image σP are related to the coördinates x_1, x_2 of P by the formulae:

$$x_1' = a_1(\sigma) + c(\sigma)x_1 - \epsilon s(\sigma)x_2,$$
$$x_2' = a_2(\sigma) + s(\sigma)x_1 + \epsilon c(\sigma)x_2,$$

where $a_1(\sigma)$, $a_2(\sigma)$, $c(\sigma)$, and $s(\sigma)$ are real numbers depending only on σ and subject to the conditions:

$$c(\sigma)^2 + s(\sigma)^2 = 1, \qquad \epsilon = \pm 1. \tag{1}$$

Among the rigid movements the most prominent are the rotations about the origin characterized by $\epsilon = +1$, $a_1(\sigma) = a_2(\sigma) = 0$, which form an abelian additive group if the addition of two rotations σ, τ is defined by applying one after the other:

$$(\sigma + \tau)P = \sigma(\tau P). \tag{2}$$

Equivalent to this fact are the addition theorems:

$$s(\sigma + \tau) = s(\sigma)c(\tau) + c(\sigma)s(\tau),$$
$$c(\sigma + \tau) = c(\sigma)c(\tau) - s(\sigma)s(\tau), \tag{3}$$

which follow from our definitions.

By application of a rotation σ about the origin to the points of the upper half-plane H:

$$x_2 > 0$$

we obtain the *open half-planes* σH. The intersection of two open half-planes is called an *angular space*. The angular spaces may be characterized as those point sets of the Cartesian plane which are open, convex, and conic with respect to the origin. An angular space is either empty or it is bounded by two open rays from the origin and the origin itself.

We define an *oriented angle* as a system consisting of an ordered pair of rays r_1, r_2 from the origin and an angular space A bounded by r_1, r_2 and the origin; it may be denoted by (r_1, r_2, A). Also, the system of two equal open rays r,r and the empty angular space Λ is defined as an oriented angle and is called a zero angle denoted by 0 or by (r,r,Λ).

We define $(r_1, r_2, A) = (r_1', r_2', A')$ if and only if there is a rotation σ so that $\sigma r_1 = r_1'$, $\sigma r_2 = r_2'$, $\sigma A = A'$, which may be written $(r_1', r_2', A') = \sigma(r_1, r_2, A)$.

We define for any three rotations ρ, σ, τ about the origin

$$\rho(r_1, r_2, A) + \sigma(r_2, r_3, B) = \tau(r_1, r_3, C)$$

provided that either $A \cap B = \Lambda$, $A \cup r_2 \cup B = C$ or $B \cap C = \Lambda$, $B \cup r_3 \cup C = A$ or $A \cap C = \Lambda$, $A \cup r_1 \cup C = B$. Furthermore: $-(r_1, r_2, A) = (r_2, r_1, A)$. Also $(r_1, r_2, A) > 0$ if there is a rotation σ so that σr_1 is the positive x_1-axis: and σA is a non-empty subset of the upper half-plane. Finally: $(r_1, r_2, A) > (r_1', r_2', A')$ if $(r_1, r_2, A) > 0$ and $(r_2', r_1', A') > 0$, or $(r_1, r_2, A) + (r_2', r_1', A') > 0$.

In other words an angle is meant to be an ordered pair of open rays from the origin together with the angular space in between. This space is uniquely determined by the two rays except for the extreme case of two opposite rays when two angles are determined: a positive angle μ and the angle $-\mu$. Due to the properties of rotations and our definitions all the axioms A, B and C can be verified.

It follows that μ is a maximum angle and $-\mu$ is a minimum angle. We remark that for two positive angles (r_1, r_2, A) and (r_1, r_3, B) the inequality $(r_1, r_2, A) > (r_1, r_3, B)$ is equivalent to $A \supset B$ (*i.e.*, B is a proper subset of A).

For a set S of positive angles $S = (r, r_S, A_S)$ the angular space belonging to the least upper bound of S is obtainable as the union of all the angular spaces A_S. The previously mentioned characterization of angular spaces may serve to prove that the union of the angular spaces A_S is an angular space used for establishing the existence of the least upper bound of S.

4. Measurement. After we have agreed on the essential properties of angles as laid down in the axioms A, B and C the problem arises as to whether angles can be measured. By this we mean the problem of setting up a 1-1 correspondence between angles and certain real numbers so that the addition and order relations are preserved:

$$\alpha \rightarrow m(\alpha)$$

so that from $\alpha + \beta = \gamma$ it follows that $m(\alpha) + m(\beta) = m(\gamma)$ and for each angle it is true that:

$$\operatorname{sgn} m(\alpha) = \operatorname{sgn} \alpha.$$

This task will be made very much easier if we agree on a suitable set of axioms describing the ordered module formed by the real numbers. In other words, once we begin to axiomatize our experiences with numbers and figures we are quite naturally guided to unifying systems of axioms which help us in our task to make transitions from one branch of mathematics to another.

As a suitable system of axioms for the real axis we can take the following:

The ordered module formed by the real numbers is characterized as a set R of at least two elements in which the addition is uniquely defined subject to the associative law: $a + (b + c) = (a + b) + c$, the law of the existence of a zero element 0 satisfying $a + 0 = a$, the existence of a negative, $-a$, for each element a of R satisfying $a + (-a) = 0$.

Furthermore for each element a of R one and only one of the three relations $a > 0$, $a = 0$, $-a > 0$ holds so that

(1) from $a>0$, $b>0$ there follows $a+b>0$;

(2) if $a+b>0$ then $b+a>0$;

(3) each subset of R bounded from above has a least upper bound, if we define, as usual, $a>b$ if and only if $a+(-b)>0$;

(4) for each positive element a there is a positive element b satisfying $a>b>0$.

The condition (3) is equivalent to

(3a) there is a fixed positive element μ of R so that every positive element of R is equal to a finite sum over positive elements each not greater than μ. Every subset of positive elements not greater than μ has a least upper bound. In this form the condition (3) can be easily verified in the course of the ensuing construction.

We notice that the preceding set of axioms describing the real module is quite similar to the system of axioms describing our concept of angle. The main difference is in the fact that two real numbers can always be added whereas two angles cannot always be added. This suggests that our angles are in 1-1 correspondence to only a part of the real axis, say an interval between the upper bound μ and the lower bound $-\mu$ or a part of it. We may prove this directly by using the idea of the Dedekind section. However, there is another way available by first extending our concept of an angle before we try to measure angles. The need for this extension is made clear by considering the purely geometrical origin of our notion of angle which does not comply with the needs of kinematics. For example, consider the system of the fast rotating record disk and the slow-moving gramophone needle. One complete revolution of the disk produces a different situation since meanwhile the needle has slightly changed its position. It does not matter whether the rotation is performed straightforwardly or in leaps and bounds, back and forth. Hence we are led to the concept of kinematical angles ranging over a wider range than the geometrical angles, which may be produced by the unrestricted addition of geometrical angles.

We define:

A *kinematical angle* is a formal sum of geometrical angles $\alpha_1, \alpha_2, \cdots, \alpha_r$ written as $\alpha_1 \dot{+} \alpha_2 \dot{+} \cdots \dot{+} \alpha_r$ or $\dot{\sum}_{i=1}^{r} \alpha_i$ in order to distinguish the formal nature of the $\dot{+}$ addition from the algebraic nature of the $+$ addition. The number r of summands ranges over the natural numbers. We define

$$\dot{\sum}_{i=1}^{r} \alpha_i = \dot{\sum}_{j=1}^{s} \beta_j$$

if the formal sum on the right-hand side can be obtained from the formal sum on the left by a finite number of *reductions* and anti-reductions. Here, a reduction means the replacement of two consecutive formal summands α_i, α_{i+1} by one formal summand $\alpha_i+\alpha_{i+1}$ provided this sum is defined. An anti-reduction is defined to be the converse of a reduction. We define the addition of kinematical angles by the rule

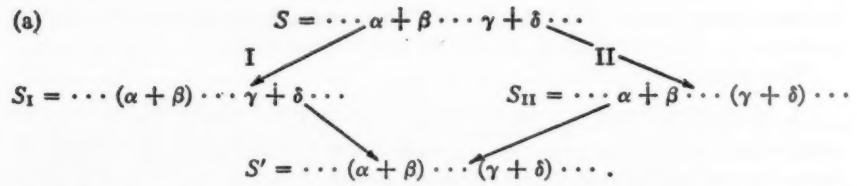
$$(\alpha_1 \dot{+} \alpha_2 \dot{+} \cdots \dot{+} \alpha_r) \dot{+} (\beta_1 \dot{+} \beta_2 \dot{+} \cdots \dot{+} \beta_s) = \alpha_1 \dot{+} \alpha_2 \dot{+} \cdots \dot{+} \alpha_r \dot{+} \beta_1 \dot{+} \beta_2 \dot{+} \cdots \dot{+} \beta_s.$$

We convince ourselves that the kinematical angles according to our definition form an additive group. As zero element we have 0 from before. The negative of the kinematical angle $\alpha_1 \dot{+} \alpha_2 \dot{+} \cdots \dot{+} \alpha_r$ is given by $-\alpha_r \dot{+} -\alpha_{r-1} \dot{+} \cdots \dot{+} -\alpha_1$. We see at once that every formal sum by mere reduction can be reduced either to 0 or to a formal sum over positive terms only or to a formal sum over negative terms only. We will use this observation in order to define an ordering of the kinematical angles by calling them 0, positive or negative corresponding to the three possible results of mere reduction just enumerated. This, however, presupposes that the result of the reduction of any formal sum equal to a given one will always lead eventually to the same type of reduced sum. We prove by induction over r:

LEMMA 1. *If a formal sum* $S = \dot{\sum}_{i=1}^{r} \alpha_i$ *can be reduced to* 0 *then every reduction eventually leads to* 0.

Let us notice that every reduction diminishes the number of formal summands by 1 so that after at most $r-1$ reductions an irreducible sum will be obtained. If $r=1$ then $\alpha_1=0$. Let $r>1$ and assume that the lemma is true for all formal sums of less than r terms which can be reduced to 0. We distinguish several cases according to the way in which the first step of a reduction, I, leading to 0 interferes with the first step of another reduction, II. We indicate by dots those terms which are not concerned by the reduction under consideration. An arrow points from a formal sum to another which is obtained from it by a reduction. We distinguish the following cases:

(a)

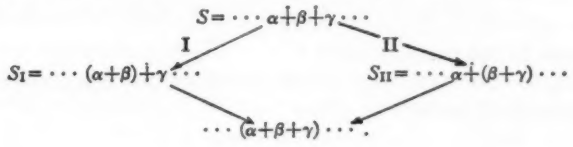


Since S_{I} has only $r-1$ terms it follows from the inductional assumption that every reduction procedure applied to S_{I} winds up with 0. Since S' can be obtained by reduction from both S_{I} and S_{II} it follows that S_{II} can be reduced to 0. Hence, by inductional assumption, every reduction procedure applied to S_{II} winds up with 0. Hence the reduction procedure II applied to S winds up with 0.

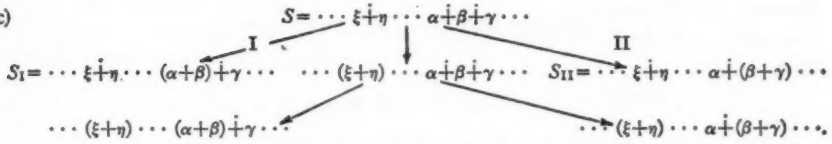
We argue similarly in the other cases which are merely outlined by inserting the links connecting S_{I} and S_{II}. In enumerating these cases we observe that the given formal sum of r terms cannot consist of merely negative terms since reduction would always lead again to formal sums of the same type so that 0 would never be obtained. Furthermore if α, β and γ are three consecutive terms

of S so that $\alpha+\beta, \beta+\gamma$ exist but neither $(\alpha+\beta)+\gamma$ nor $\alpha+(\beta+\gamma)$ exists, then either α, β and γ are all positive or all negative. This is because none of the three angles α, β, γ vanishes. Also if β and γ are of the same signature, α of opposite signature, then $\beta+\gamma$ will be of the same signature as β and γ and hence $\alpha+(\beta+\gamma)$ will exist contrary to our assumption. Similarly it follows that it is impossible that α and β have the same and γ the opposite signature.

(b)

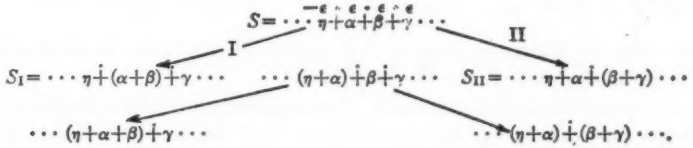


(c)

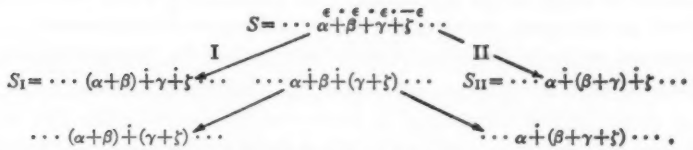


In cases (d) and (e) the ϵ's represent the signatures of the corresponding angles.

(d)



(e)



Other cases, apart from the trivial case $S_I = S_{II}$ and complete reversal of the order of addition do not occur.

LEMMA 2: *If a formal sum* $S = \sum_{i=1}^{r} \alpha_i$ *is equal to* 0 *then it can be reduced to* 0.

We apply induction over the number, n, of reductions and anti-reductions which are necessary to carry S over into 0.

For $n=1$ the lemma is clear.

The first step leads to a formal sum S' which according to the inductional assumption can be reduced to 0. We have either $S \to S'$ in which case S can be reduced to 0 obviously or $S' \to S$ in which case S can be reduced to 0 by lemma 1.

Let sgn $\beta = 1$. If sgn $\alpha = 1$ then $\beta + -\alpha$ exists as well as $-\alpha+\alpha$. Then, $\beta+(-\alpha+\alpha)=\beta+0=\beta$, hence also $(\beta+-\alpha)+\alpha=\beta$ and $\gamma=\alpha+(\beta+-\alpha)$ is defined and has the same signature as β. If sgn $\alpha=-1$ then $\alpha+\beta$, $-\alpha+\alpha$ exist,

$(-\alpha+\alpha)+\beta=0+\beta=\beta$ so that also $-\alpha+(\alpha+\beta)=\beta$ and $\gamma=(\alpha+\beta)+-\alpha$ exists and has the same signature as β. It follows that $\alpha\dot{+}\beta\dot{+}-\alpha$ can be reduced to γ and hence is positive. Repeated application of this argument yields that

$$\dot{\sum}_{i=1}^{r}\alpha_i \dot{+} \beta \dot{+} \sum_{i=r}^{1} - \alpha_i$$

$$= \alpha_1 \dot{+} \alpha_2 \dot{+} \cdots \dot{+} \alpha_r \dot{+} \beta \dot{+} - \alpha_r \dot{+} - \alpha_{r-1} \dot{+} \cdots \dot{+} - \alpha_1$$

can be reduced to one positive term.

Since $S\dot{+}\beta\dot{+}-S=-(S\dot{+}-\beta\dot{+}-S)$, it follows that sgn $(S\dot{+}\xi\dot{+}-S)=$sgn ξ for every geometrical angle ξ.

If

$$\operatorname{sgn}\beta_1 = \operatorname{sgn}\beta_2 = \cdots = \operatorname{sgn}\beta_q = \epsilon$$

then

$$\operatorname{sgn}(S \dot{+} \beta_j \dot{+} - S) = \operatorname{sgn}\beta_j = \epsilon$$

$$\epsilon = \operatorname{sgn}(\dot{\sum}(S \dot{+} \beta_j \dot{+} - S)) = \operatorname{sgn}(S \dot{+} \dot{\sum}\beta_j \dot{+} - S)$$

$$\operatorname{sgn}(S \dot{+} T \dot{+} - S) = \operatorname{sgn} T$$

$$\operatorname{sgn}(S \dot{+} T) = \operatorname{sgn}(-S + (S + T) + - (-S)) = \operatorname{sgn}(T + S).$$

It follows that the additive group A of kinematical angles can be ordered by the rule: $T>S$ if and only if $T\dot{+}-S$ is positive.

The kinematical angle corresponding to a given geometrical angle α has the same signature as α. Furthermore it follows that the correspondence between the geometrical angle α and the kinematical angle α is 1-1 and preserves addition as well as signature. In other words we have succeeded in extending the set of geometrical angles to the ordered additive group of the kinematical angles.

If $0<s<\alpha$ then $s=\dot{\sum}_{i=1}^{r}\alpha_i$, $\alpha\dot{+}-S=\dot{\sum}_{j=1}^{q}\beta_j$ with sgn $\alpha_i=$sgn $\beta_j=1$,

$$-S = -\alpha \dot{+} \dot{\sum}_{j=1}^{q}\beta_j = (-\alpha + \beta_1) \dot{+} \beta_2 + \cdots + \beta_q$$

$$= ((-\alpha + \beta_1) + \beta_2) \dot{+} \beta_3 + \cdots \dot{+} \beta_q = \cdots.$$

Hence every positive kinematical angle which is not greater than a geometrical angle is itself geometrical. Denoting by μ the least upper bound of all geometrical angles we verify the condition 3a for the ordering of real numbers.

If there is no smallest positive geometrical angle, then the ordered additive group A of the kinematical angles satisfies all the axioms of the real module R. Since these axioms characterize R up to an order preserving isomorphism it follows that there is an order preserving isomorphism m of A onto R. In other words there is a measurement of the kinematical angles. This leads also to a measurement of the geometrical angles.

The order preserving isomorphism m is uniquely determined up to an order

preserving automorphism of R which is obtained by the multiplication of each real number by a fixed positive number. In other words, the measurement of angles is unique up to a positive factor of proportionality. A normalization is obtained by prescribing the measure $m(\mu)$. This must be a positive real number.

The Babylonians introduced $m(\mu)=180$ because of the relatively many divisors of this number. The artillery uses $m(\mu)=3200$ because of the high power of 2 which divides it as well as a suitable power of 10. It has been suggested to set $m(\mu)=\frac{1}{2}$ corresponding to $m(2\mu)=1$ for the full circuit but this convention is nowhere in use. Instead of taking 180 degrees many instruments are graduated into 200 degrees corresponding to $m(\mu)=200$. This convention assigns 100 degrees to the quadrant which is in accordance with the general tendency to introduce the decimal system everywhere. Finally, the transcendental number $\pi=3.14159\cdots$ has been widely used for the normalization of $m(\mu)$. This can be derived from the local property:

$$\text{l.u.b.}_{\alpha>0}\ \frac{s(R_\alpha)}{m(\alpha)}=1,$$

where R_α denotes the rotation about the origin through the angle α.

For this measure we define

$$\begin{aligned}\sin m(\alpha) &= s(R_\alpha)\\ \cos m(\alpha) &= c(R_\alpha).\end{aligned}\tag{3}$$

We convince ourselves that the map

$$S=\sum_{i=1}^{r}\alpha_i\rightarrow R_{\alpha_1}+R_{\alpha_2}+\cdots+R_{\alpha_r}=R_s$$

is a homomorphic map of the group $\mathfrak{A}$ of the kinematical angles onto the group $\mathfrak{R}$ formed by the rotations about the origin. This must be so because the reductions and anti-reductions in $\mathfrak{A}$ correspond to identical substitutions in $\mathfrak{R}$.

Hence we may extend (3) by

$$\begin{aligned}\sin m(s) &= s(R_s)\\ \cos m(s) &= c(R_s).\end{aligned}\tag{4}$$

We then obtain the addition theorems and, in fact, all essential properties of the trigonometric functions in terms of a suitable angular measure.

If there is a smallest positive angle, say μ_0, then it follows by a well known argument that every angle is a multiple of μ_0. In this case all angles are commensurable, which is of course the only case realizable in practice. Again we find that, up to a constant factor of proportionality λ, there is only one angular measurement.

It is realized as $m(g\mu_0)=\lambda g(g=0,\ \pm1,\ \pm2,\ \cdots,\ \lambda>0)$. The corresponding plane trigonometry would be related to the regular polygon of $2n$ sides and vertices where $\mu=n\mu(0)$.

I would just point out that our construction of the module formed by the kinematical angles which has them generated by the geometrical angles is closely related to the construction of a covering group by Otto Schreier. In fact we can use our construction for a proof of the well known theorem that every 1-parameter group germ generates the additive group of the real numbers after suitable choice of the parameter. This proof avoids entirely the extraction of square roots; apart from our construction only some simple theorems on continuous functions need be used.

5. Conclusion. In closing this paper I can say this much in favor of the axiomatic method: it forces us to concentrate on the first principles and gives us back that humble attitude towards the unknown which is in constant danger of being lost under the weight of too many detailed results which we are often compelled to swallow whole. However, the axiomatic approach formally appears merely as a nice way of organizing things unknown and things known. The upshot of these two statements is that the axiomatic treatment is, as is everything else in the development of our science, the fruit of a particular historical situation, and that there is no fear that yet another historical situation may provide us with new points of view amplifying and revitalizing the axiomatic approach. Nor need you fear that the axiomatic treatment endangers your originality. It is entirely up to you to suit yourself with such foundations in a particular branch of mathematics as are most readily adaptable to your own problems. For example, I am quite sure that at some future time another mathematician will turn up and give his ideas on what an angle ought to be; ideas which will be quite different from those given today. His only obligation will be that his conception comprehends those concepts of angle and angular measurement which have been previously adopted and I might say it would be nice of him to extend our knowledge in the process. So why then are we doing this type of mathematics so far exceeding the necessity of making computations for daily use? Is it because we learn the principles governing effective computations and thereby do them better? Or because of the beauty of the logical pattern which evolves? Or because we desire to get a more powerful grip of the strings by which the isolated facts of mathematical experience are related? Or, simply because we must?

COMMENTS ON MATHEMATICAL EDUCATION*

J. W. LASLEY, JR., University of North Carolina

The need for closer cooperation between high school, college, and university teachers is receiving more attention at the present time than perhaps at any time in the past. Recognition of this need is not confined to the teaching profes-

* Written in December, 1953.

sion but is definitely high in the thinking of our government and industry. Essentially, the basic objective here is to find students with capacity for mathematical training while they are in the secondary schools, give them proper training and guidance throughout a program of study leading through the university, and thereby supply the nation with an adequate number of capable and well-trained mathematicians. Efforts to meet the need just cited are being made, in varying degrees of adequacy, by many educational institutions in the country. It is thought that reports on several of these might serve a number of good purposes, among which are: provide basis for over-all appraisal of the extent to which the need is being met, suggest variations in approaches being used in a given locality, lead to a closer cooperation between institutions that are seriously concerned about the matter. The present note reports some of the recent approaches used at the University of North Carolina.

1. High school mathematics contest. The University holds each year a high school mathematics contest with awards for the winner and recognition by honorable mention for the runners-up. The local Pi Mu Epsilon chapter makes an award to the winner. It is contemplated that scholarships be provided for this meritorious achievement. A thousand high school seniors participate in this contest. On High School Day when many hundreds of high school students come to the University, this contact with the schools of the State is greatly broadened. Opportunity is provided for meeting the mathematics staff, seeing the places where future university students will study, looking at models, exhibits, *etc.*, tending to quicken their interest in mathematical studies.

2. Contacts between high school and university teachers. The North Carolina Education Association has a mathematics section. Its meeting occurs concurrently with that of the larger body. It is felt here that a more specific get-together of this group with the University would be mutually beneficial. Consequently, this year such a meeting is projected for the first week in March.* It is to be a weekend session with both formal talks and small discussion groups. Three of these sub-groups will meet for afternoon, night, and morning-after sessions. Each will have one particular phase of high school-college relationship subjected to common scrutiny. It is hoped that this program, sparked by two or three formal addresses, will attract a hundred or more high school teachers.

The University is seeking a closer cooperation with the State Department of Public Instruction in the area of mathematics. In the early spring of this year an interested group from our mathematics staff met with a representative of that Department with a view to better understanding of our mutual problems at the high school teaching level. This representative proved to be eager, sensitive, and aware of many of these common problems. We sat down at table and talked these matters out at length. We decided to send a selected few of our staff to some of the meetings which this representative holds periodically with the county groups of high school mathematics teachers. In a few weeks a start

was made on this project, and it is our intent to follow up with similar meetings in many sections of the State. We found the teachers in these county centers entirely cooperative and interested. There was no spirit of recrimination evident. We met on the basis of a common job to do and a mutual sharing of the difficulties which it presented. We, at any rate, found these associations worthwhile.

3. Distribution of mathematical information. The General Electric Company of Schenectady, New York, has issued recently, in addition to its bimonthly News Digest, two pamphlets: one entitled "Why Study Math?," the other "Math at General Electric." These brochures are in the modern tempo, with direct appeal, jazzed up with cartoons. Few high school students could read them unmoved. These pamphlets were featured in the regular department bulletin board exhibits of interesting current mathematical materials. There was sent out in September of this past year the first of these to teachers of mathematics in all the colleges and high schools of the State. Copies of the second of these pamphlets are now in process of being similarly mailed. The potential for interest in this contact is believed to be considerable.

4. Special courses and fellowships. For some time now the mathematics courses offered in the summer session have been directed in a substantial part to high school teachers. Last summer the offerings in algebra designed for teachers were revamped into a course in special topics in modern algebra. This was part of a concerted move on the part of mathematics together with the several sciences to give an interlocking program to insure a full program for the teacher-student. During the ensuing summer this course in algebra is to be re-offered with alterations. A somewhat similar course in geometry will be added. This coming summer all courses are to be set up on an 80-minute 5-day schedule, with two such courses constituting a full-time load.

The Du Pont Company of Wilmington, Delaware, upon learning of the efforts being made at this university for better high school-college relationships, set up through the School of Education twelve Du Pont Fellowships for the summer session of 1954. A committee of representatives of the mathematics department and the several science departments together with the Dean of the School of Education are administering this fellowship program. It is heartening to observe that the cooperation between the Dean and the department representatives is all one could desire. Each of these fellowships carries a compensation of $225 in addition to coverage of tuition and fees. The tenure is for twelve weeks. The work is to be centered within subject matter areas, at least one of which must be mathematics or a science. The awards are not restricted to this state.

DOODLES

G. B. ROBISON, University of Connecticut

1. To the mathematically minded doodler, the fact that there is an interrelation between doodling and the other topics of this paper is, to coin a phrase, self-evident. There are even those who would assert that all of pure mathematics is merely doodling according to an arbitrary set of rules. Nor should it amaze one to find a connection between space-filling curves and Cantor Sets, since the latter are at the heart of more than one pathological specimen.

The particular doodle I have in mind may be classified as a multiflora rose doodle. Starting with an arbitrary convex polygon, extend one of the sides to a convenient point. Then draw a line from this point through the next vertex, extending it to a position suitable for continuation, doodle-wise (Fig. 1).

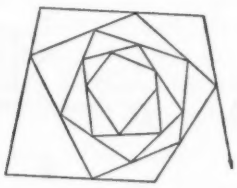


FIG. 1

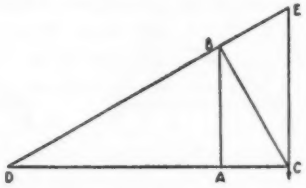


FIG. 2

If we start with a triangle ABC, the question may be raised as to how we can proceed to make the added triangles each similar to the original one. It is quickly seen that the vertex from which the first extension is made must be at a right angle (Fig. 2).

Let us now take Jacobi's advice and invert the construction. Starting with a right triangle ABC (Fig. 3), draw the altitude CD, then the altitude DE of $\triangle ACD$, then EF in $\triangle ECD$, etc., always taking clockwise turns. The sequence of points $C, D, E, F \cdots$ converges to a point P.

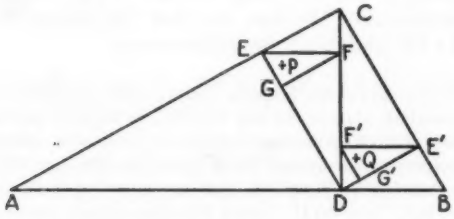


FIG. 3

If we take counterclockwise turns, $C, D, E', F' \cdots$ converge to a point Q. In $\triangle ACD$, the points $D, E, F, G \cdots P$ are homologous to $C, D, E', F' \cdots Q$

in $\triangle ABC$, and $D, E', F', G' \cdots Q$ in $\triangle BCD$ are homologous to $C, D, E, F \cdots P$ in $\triangle ABC$. We shall discuss P and Q again under more general conditions.

2. The fact that a curve can be continuously mapped onto a region of higher dimension was foreshadowed by Cantor's proof of the cardinal equivalence of such point sets.* The one-one mappings used in establishing cardinal equivalence are of necessity discontinuous.† Another interesting mapping of a line segment onto a square is given by Cesaro.‡ This has the interesting property that as t varies through any interval on the line, $f(t)$ takes the value of a predetermined point in the square an infinite number of times. The first continuous mappings of a line segment onto a square were exhibited by Peano in 1890, and Hilbert in 1892.§ A third curve due to Polya†† fills a right triangle, and this one is related to the inverse doodle.

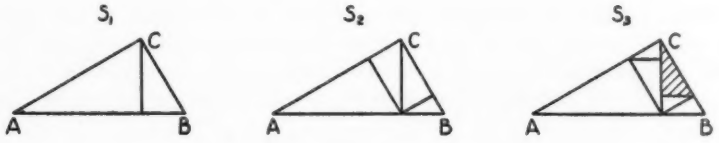


FIG. 4

Let I be the interval $0 \leqq t \leqq 1$. Let ABC be a scalene‡‡ right triangle. The altitude to the hypotenuse divides the triangle into two triangles, each similar to the first. Call this figure S_1. Divide each subtriangle again, obtaining S_2, and so on. The sequence $\{S_i\}$ represents successive approximations to values of $f(t)$ for $t \in I$, expressed in dyadic notation, as follows: If the first digit of t is 0 assign $f(t)$ to the smaller triangle in S_1, if 1, to the larger. If the second digit is 0, assign it to the corresponding smaller triangle in S_2, if 1, to the larger, *etc.* For example, the shaded triangle in S_3 represents $f(t)$ for all values of t which begin .011. Thus for each value of t we have a sequence of nesting triangles with diameter approaching zero, and hence converging on a unique point. The ambiguity of pairs like .1$\dot{0}$ and .0$\dot{1}$ does not cause trouble as the corresponding sequences converge on the same point. We now see that the points P and Q of Figure 3 correspond to 1/3 (.0$\dot{1}$) and 2/3 (.$\dot{1}\dot{0}$) respectively.

* See in particular Journal für Mathematik, vol. 84, 1878, pp. 242 ff.

† See H. Hahn, Annali di Matematica vol. 21, 1913, in which is proved that any continuous mapping of a line segment onto a square has triple image points dense in the square.

‡ Bulletin des Sciences Mathematique, Series 2, vol. 21, 1897, pp. 257–266.

§ Mathematische Annalen V. 36 and 38. Both are described in Pierpont "Theory of Functions of a Real Variable" V.2, Chap. XVII. A good discussion of both curves as well as the notorious Weierstrass curve and other "crinkly" curves is found in an article by E. H. Moore in the A.M.S. Transactions, Vol. I, 1900.

†† This description is adapted from that given by S. Lefschetz in "Introduction to Topology," pp. 5, 6.

‡‡ For convenience of description.

Let us now examine which points of I map into the hypotenuse.

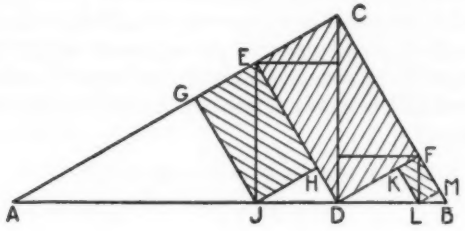


FIG. 5

The rectangle $ECFD$ is excluded (except point D), and this corresponds to the values $1/4<t<3/4$. Likewise in the triangles remaining, we must exclude the rectangles $GEHJ$ and $KFML$ (excepting once more the points J and L) and these areas correspond to values $1/16<t<3/16$ and $13/16<t<15/16$. We are excluding the (open) middle halves of the remaining intervals of I at each step. The end result is a Cantor Set.

3. An interesting extension of this idea is found by taking any triangle having a largest angle and drawing *two* lines from the vertex of this angle. In Figure 6, $\sphericalangle ACD=\sphericalangle ABC$ and $\sphericalangle ECB=\sphericalangle BAC$ and hence $\triangle ACD\sim\triangle ABC\sim\triangle CBE$.

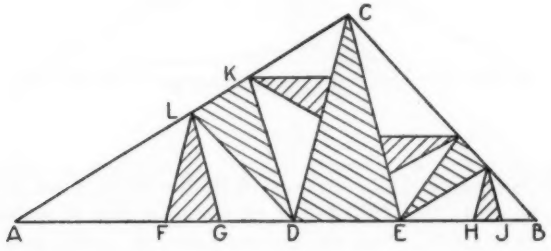


FIG. 6

This process can be repeated in turn with each of the smaller triangles, and I can be mapped into the given triangle as before.

Let us examine the distribution of image points for the case where $\sphericalangle C>90°$. At the first step, the open interval DE is excluded. At the third step, open intervals FG and HJ are excluded, and we see the Cantor Set emerging on the other side of the mapping. Each bounding line (*e.g.*, AC, CD, LD) is in turn similarly dissected and we have a network of Cantor Sets in $f(I)$. This does not constitute the whole image, but solely those values of $f(t)$ for which t, expressed in notation to the base 4, contains only a finite number of occurrences of the digits 1 and 2. In fact the complete image is a homeomorph of I! As might almost be expected,

it has no tangents anywhere, and if one extracts the functions $x(t)$ and $y(t)$, one will find that they are nowhere differentiable.*

It is not necessary that $\sphericalangle C$ in Figure 6 be obtuse (or even the largest angle) to get convergence of the particular triangle sequences for $f(1/3)$ and $f(2/3)$. One can show that these triangle sequences will converge if, and only if, $c > \sqrt{ab}$. In particular, if one of the base angles (say $\sphericalangle A$) is a right angle, then in the limiting case we have:

$$a^2 - b^2 = c^2 = ab$$

$$a^2 - ab = b^2$$

$$\frac{a}{b} = \frac{b}{a-b},$$

for which the Greeks had a word!

4. The property of P and Q, that P in $\triangle ACD$ is homologous to Q in $\triangle ABC$ and Q in $\triangle CBE$ is homologous to P in ABC, is independent of any consideration of convergence of the mapping. In Figure 7 we have taken C as the smallest angle, in which case no triangle sequence converges. P and Q still exist in $\triangle ABC$, in fact they are the Brocard points!†

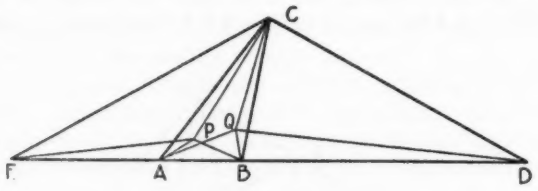


FIG. 7

THEOREM: *On triangle ABC, construct $\sphericalangle BCE = \sphericalangle BAC$ and on the same side of BC as A, and $\sphericalangle ACD = \sphericalangle CBA$ and on the same side of AC as B, forming $\triangle CBE \sim \triangle ACD \sim \triangle ABC$. Then the Brocard points of $\triangle ABC$ have the similarity property described above, and they are the only pair having this property.*

Remark. Considerations of order are satisfied by:

1. $\sphericalangle A + \sphericalangle B < 180°$, and hence E will lie on the same side of B as A, and D on the same side of A as B.

2. The Brocard angle is less than any angle of the triangle, hence, P will always be inside $\triangle CBE$ and Q inside $\triangle ACD$.

3. We will follow the convention that $\sphericalangle XYZ = -\sphericalangle ZYX$ and hence

* The same is true of Hilbert's and Peano's curves. See E. H. Moore *loc. cit.*

† I have not found this property of the Brocard points mentioned elsewhere. I am indebted to Professor R. J. Walker for the suggestion that these might be the Brocard points, because of the rarity of symmetrically related pairs of points in the triangle.

$\sphericalangle WXY + \sphericalangle YXZ$ always equals $\sphericalangle WXZ$. In applying this, it should be noted that $\triangle ABC$ is reversed relative to $\triangle CBE$ and ACD. Thus, for example, $\sphericalangle CBE = \sphericalangle ACD = -\sphericalangle ABC$. (Order of corresponding vertices has been maintained in naming similar triangles.)

I. Let P and Q be Brocard points of $\triangle ABC$, with $\sphericalangle PAC = \sphericalangle PCB = \sphericalangle PBA = \sphericalangle BAQ = \sphericalangle CBQ = \sphericalangle ACQ$. We will show that $\triangle AQC$ (in $\triangle ACD$) is similar to $\triangle APB$ (in $\triangle ABC$):

$$\sphericalangle QAC = \sphericalangle QAB + \sphericalangle BAC = \sphericalangle CAP + \sphericalangle BAC = \sphericalangle BAP,$$
$$\sphericalangle ACQ = \sphericalangle PBA,$$

therefore

$$\triangle AQC \sim \triangle APB.$$

Similarly we can show $\triangle CPB$ (in $\triangle CBE$) $\sim \triangle AQB$ (in $\triangle ABC$).

II. Assume that P and Q are points in $\triangle ABC$ having the similarity property. We will show that $\sphericalangle PCB = \sphericalangle PAC = \sphericalangle PBA$.

$$\triangle AQC \text{ (in } \triangle ABC) \sim \triangle CPE \text{ (in } \triangle CBE),$$

and

$$\triangle AQC \text{ (in } \triangle ACD) \sim \triangle APB \text{ (in } \triangle ABC);$$

therefore

$$\sphericalangle PAB = \sphericalangle CAQ = \sphericalangle PCE;$$

but

$$\sphericalangle BAC = \sphericalangle ECB,$$

hence

(A) $$\sphericalangle PAC = \sphericalangle PAB + \sphericalangle BAC = \sphericalangle PCE + \sphericalangle ECB = \sphericalangle PCB.$$

Furthermore,

$$\sphericalangle BPA = \sphericalangle EPC \qquad \text{(similarity of triangles)},$$

and therefore

$$\sphericalangle BPE = \sphericalangle BPA + \sphericalangle APE = \sphericalangle EPC + \sphericalangle APE = \sphericalangle APC.$$

From

$$\frac{BP}{PA} = \frac{EP}{PC} \qquad \text{(similarity of triangles)},$$

we get

$$\frac{BP}{PE} = \frac{AP}{PC},$$

hence

$$\triangle BPE \sim \triangle APC,$$

and

(B) $$\sphericalangle PAC = \sphericalangle PBE.$$

(A) and (B) are the desired result. Similarly we can show that

$$\sphericalangle QBC = \sphericalangle QCA = \sphericalangle QAB.$$

THE ELEMENTARY TRANSCENDENTAL FUNCTIONS

W. F. EBERLEIN, University of Wisconsin

It is difficult to differentiate the undefined.

1. Introduction. We outline a unified approach to the circular and hyperbolic functions motivated by the traditional geometric considerations (Figs. 1a and 1b), yet leading immediately to a sound analytic definition. The treatment in whole or part seems adaptable to modern calculus courses in which integration appears early and some knowledge of trigonometry is presupposed.

2. Geometric approach to the circular and hyperbolic functions. Consider the unit circle (right branch of the equilateral hyperbola)

(1a) $x^2 + y^2 = 1$

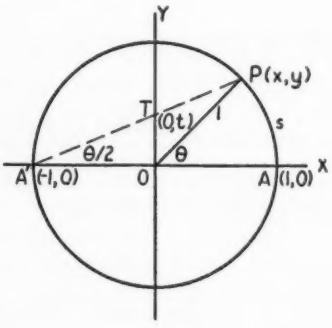


FIG. 1a

(1b) $x^2 - y^2 = 1$

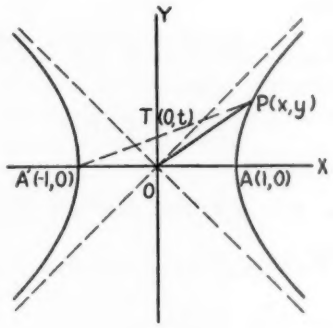


FIG. 1b

Letting θ denote twice the signed area of the sector OAP, regard the coordinates (x, y) of the variable point P as functions of θ:

(2a) $x = \cos\theta,$ $y = \sin\theta.$ | (2b) $x = \cosh\theta,$ $y = \sinh\theta.$

[$\theta = s = \widehat{AP}$ is then the radian measure of $\angle AOP$.]

In particular

(3a) $\cos 0 = 1,$ $\sin 0 = 0.$ | (3b) $\cosh 0 = 1,$ $\sinh 0 = 0.$

It follows from Figure 1 that in both cases

$$\theta = xy - 2\int_1^x y\,dx. \tag{4}$$

Taking differentials in equations (1) and (4) yields

(5a) $0 = x\,dx + y\,dy,$ | (5b) $0 = x\,dx - y\,dy.$

$$d\theta = x\,dy + y\,dx - 2y\,dx = x\,dy - y\,dx. \tag{6}$$

Now solve equations (5) and (6) for the unknowns dx and dy:

(7a) $dx = -y\,d\theta,$ $dy = x\,d\theta.$ | (7b) $dx = y\,d\theta,$ $dy = x\,d\theta.$

Hence

(8a) $D_\theta \cos\theta = -\sin\theta,$ $D_\theta \sin\theta = \cos\theta.$ | (8b) $D_\theta \cosh\theta = \sinh\theta,$ $D_\theta \sinh\theta = \cosh\theta.$

To justify the formal differentiation consider the line $A'P$ joining $A':(-1, 0)$ and the point $P:(x, y)$, where $P \neq A'$. Let $A'P$ intersect the y axis in the point $T:(0, t)$. It follows from the figure that

(9a) $t = \tan\dfrac{\theta}{2},$ | ?

$$t = y/(1 + x). \tag{10}$$

Solving for the intersections of the line $A'T$ (10) and the curves (1) yields for the coordinates of P:

(11a) $x = \dfrac{1 - t^2}{1 + t^2},$ $y = \dfrac{2t}{1 + t^2}$ $(-\infty < t < \infty)$ | (11b) $x = \dfrac{1 + t^2}{1 - t^2},$ $y = \dfrac{2t}{1 - t^2},$ $-1 < t < 1).$

Equations (11) define a one-to-one differentiable parametric mapping of the open interval $-\infty<t<\infty\,(-1<t<1)$ onto the unit circle minus the point A' (the right branch of the rectangular hyperbola):

(12a)
$$dx=-\frac{4t}{(1+t^2)^2}\,dt,\qquad dy=\frac{2(1-t^2)}{(1+t^2)^2}\,dt.$$

(12b)
$$dx=\frac{4t}{(1-t^2)^2}\,dt,\qquad dy=\frac{2(1+t^2)}{(1-t^2)^2}\,dt.$$

Since (4) now defines θ as a differentiable function of t, equations (5) and (6) are justified. Substitute the values (11) and (12) into (6) to obtain*

(13a)
$$d\theta=\frac{2dt}{1+t^2},$$

(13b)
$$d\theta=\frac{2dt}{1-t^2}.$$

Note that equations (7) and (8) follow directly from (11), (12), and (13).

Since $\theta=0$ when $t=0$,

(14a)
$$\theta=2\int_0^t\frac{du}{1+u^2}\quad(-\infty<t<\infty),$$

(14b)
$$\theta=2\int_0^t\frac{du}{1-u^2}\quad(-1<t<1).$$

The equations (9a), (11a), and (13a) are the familiar substitutions used to integrate rational functions of $\sin x$, $\cos x$. Moreover, since it follows from Figure 1a that $\theta=\frac{1}{2}\pi$ when $t=1$ and $\theta\to\pi$ when $t\to\infty$,

(15)
$$\frac{\pi}{4}=\int_0^1\frac{du}{1+u^2},\qquad\frac{\pi}{2}=\int_0^\infty\frac{du}{1+u^2}.$$

To obtain an analytic theory of the circular and hyperbolic functions we need only retrace our steps carefully.

3. Analytic theory of the circular functions. Define π by the first of equations (15). The second equation then follows on making the change of variable $u=v^{-1}$:

$$\int_1^\infty\frac{du}{1+u^2}=\lim_{t\to\infty}\int_1^t\frac{du}{1+u^2}=\lim_{t\to\infty}\int_{t^{-1}}^1\frac{dv}{1+v^2}=\int_0^1\frac{dv}{1+v^2}.$$

Equation (14a) defines θ as an odd increasing function of $t\,(-\infty<t<\infty)$ with positive derivative (13a). Invert (14a) to obtain t as an odd increasing differentiable function of $\theta\,(-\pi<\theta<\pi)$. Equations (11a), (2a) now define $\cos\theta$, $\sin\theta$ as

* If θ is defined by the arc length s, (13a) appears as a consequence of the relation $ds^2=dx^2+dy^2$.

functions of θ over the open interval $-\pi<\theta<\pi$.† Cos θ, sin θ extend to continuous functions over the closed interval $-\pi\leqq\theta\leqq\pi$ on setting

$$\cos(\pm\pi) = \lim_{\theta\to\pm\pi}\cos\theta = \lim_{t\to\pm\infty}\frac{1-t^2}{1+t^2} = -1,$$

$$\sin(\pm\pi) = \lim_{\theta\to\pm\pi}\sin\theta = \lim_{t\to\pm\infty}\frac{2t}{1+t^2} = 0.$$

It is then legitimate to extend cos θ, sin θ to arbitrary real values of θ by the requirement

$$\begin{aligned}\sin(\theta+2\pi) &= \sin\theta,\\ \cos(\theta+2\pi) &= \cos\theta.\end{aligned}\tag{16a}$$

The relations $\cos(-\theta)=\cos\theta$, $\sin(-\theta)=-\sin\theta(-\pi<\theta<\pi)$ resulting from (11a) are clearly preserved under extension.

Consider now the differentiation formulae (8a). For the interval $-\pi<\theta<\pi$ they follow again from (11a), (12a), and (13a). Their validity for $\theta=\pm\pi$ results from a familiar application of the law of the mean: For example, $\cos\pi-\cos\theta=-(\pi-\theta)\sin\xi$ $(\theta<\xi<\pi)$; whence the left derivative

$$\cos'(\pi)=\lim_{\theta\to\pi}(\cos\pi-\cos\theta)/(\pi-\theta)=-\lim_{\xi\to\pi}\sin\xi=0$$

exists and has the correct value. The validity of (8a) for unrestricted values of θ then follows from the periodicity condition (16a).

Consider the remaining equations. (3a) and (10) follow directly from (11a); (1a) requires an additional continuity and periodicity argument. The equations (5a), (6a), (7a), (12a), (13a) leading from (4) to (14a) were previously justified, whence (4) and (14a) define the same quantity θ.* Note finally that the relation (9a) follows from (10) via the double angle corollaries of the addition formulae:

LEMMA:

$$\begin{aligned}u(\theta) &\equiv \sin(\theta+\theta_0)-\sin\theta\cos\theta_0-\cos\theta\sin\theta_0=0;\\ v(\theta) &\equiv \cos(\theta+\theta_0)-\cos\theta\cos\theta_0+\sin\theta\sin\theta_0=0.\end{aligned}\tag{19a}$$

Proof: (8a) implies that $u'(\theta)=v(\theta)$, $v'(\theta)=-u(\theta)$, whence $D_\theta(u^2+v^2)=2uv-2uv=0$. Hence $u^2+v^2\equiv[u(0)^2+v(0)^2]=0$ by (3).

4. Analytic theory of the hyperbolic and exponential functions. The development of the hyperbolic functions is even simpler in that no continuity or

† $t=1$ or $\theta=\pi/2$ is then the smallest positive zero of $x(t)=\cos\theta$. This result is usually taken as the definition of π in the power series approach to the circular functions.

* Changing to the variable t in (4) leads to an integral which is difficult to transform directly into (14).

periodicity arguments are required. Equation (14b) defines θ as an odd increasing function of $t(-1<t<1)$ with positive derivative (13b). That $\theta \uparrow \infty$ when $t \uparrow 1$ follows readily from inequalities of the type

$$\int_{1-1/n}^{1-1/(2n)} \frac{du}{(1+u)(1-u)} > \frac{1}{2}\int_{1-1/n}^{1-1/(2n)} \frac{du}{1-u} \geqq \frac{1}{2}\int_{1-1/n}^{1-1/(2n)} \frac{du}{n^{-1}} = \frac{1}{4}.$$

Invert (14b) to obtain t as an odd increasing differentiable function of θ $(-\infty<\theta<\infty)$. Equations (11b), (2b) then define $\cosh\theta$, $\sinh\theta$ for *all* real values of θ, and $\cosh(-\theta)=\cosh\theta$, $\sinh(-\theta)=-\sinh\theta$. Equations (8b) and the remaining equations now follow even more readily than before.

To obtain the missing equation (9b) introduce the function

$$E(\theta) = \cosh\theta + \sinh\theta = x + y = \frac{1+t}{1-t} > 0 \qquad (-1<t<1), \tag{17b}$$

and observe that

$$\begin{aligned} E'(\theta) &= E(\theta), \\ E(0) &= 1. \end{aligned} \tag{18b}$$

Moreover, $E(\theta)\uparrow\infty$ when $\theta\uparrow\infty$ $(t\uparrow 1)$, while $E(\theta)\downarrow 0$ when $\theta\downarrow-\infty$ $(t\downarrow -1)$.

Now set $e=E(1)$ and identify $E(\theta)$ with e^θ in standard fashion via the addition formula:

$$E(\theta_1+\theta_2) = E(\theta_1)E(\theta_2). \tag{19b}$$

LEMMA: *Let $E(\theta)$ be any differentiable function of θ satisfying the equations $E'(\theta)=aE(\theta)$ $(-\infty<\theta<\infty)$ for some constant a, and $E(0)=1$. Then $E(\theta)$ satisfies* (19b).

Proof: Set $u(\theta)=E(\theta+\theta_0)E(-\theta)$. Then $u'(\theta)=aE(\theta+\theta_0)E(-\theta)-aE(\theta+\theta_0)\cdot E(-\theta)=0$. Hence $u(\theta)\equiv u(0)$, or $E(\theta+\theta_0)E(-\theta)=E(\theta_0)$. The symmetric form (19b) results on writing $\theta_0=\theta_1+\theta_2$, $\theta=-\theta_2$.

Noting finally that (17b) implies the familiar formulae

$$\begin{aligned} \cosh\theta &= \tfrac{1}{2}(e^\theta + e^{-\theta}), \\ \sinh\theta &= \tfrac{1}{2}(e^\theta - e^{-\theta}), \end{aligned} \tag{20b}$$

solve(17b) for t to obtain

$$t = \frac{e^\theta - 1}{e^\theta + 1} = \frac{\tfrac{1}{2}(e^{\theta/2} - e^{-\theta/2})}{\tfrac{1}{2}(e^{\theta/2} + e^{-\theta/2})} \equiv \tanh\frac{\theta}{2}. \tag{9b}$$

The function $w=\ln v$ $(v>0)$ may be defined in elementary fashion by the equation $v=e^w\cdots$.

5. Euler's formula and the Fourier constant. The fundamental relations between the circular and hyperbolic functions are just the missing analogues

(16b), (17a), (18a), (20a). Under the formal changes of variable $t=-is$, $u=-iv$, where $i=\sqrt{-1}$, (14a) and (11a) become:

$$i\theta = 2\int_0^s \frac{dv}{1-v^2},$$

$$\cos\theta = \frac{1+s^2}{1-s^2} = \cosh i\theta,$$

$$\sin\theta = -i\frac{2s}{1-s^2} = -i\sinh i\theta.$$

The relation $e^{i\theta}=\cosh i\theta+\sinh i\theta$ then yields Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta = \frac{1+it}{1-it}. \tag{17a}$$

Now take (17a) as a definition and observe that

$$e^{i(\theta+2\pi)} = e^{i\theta}; \tag{16b}$$

$$D_\theta e^{i\theta} = ie^{i\theta}, \tag{18a}$$

$$e^{i0} = 1;$$

$$\cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta}). \tag{20a}$$

$$\sin\theta = \frac{1}{2i}(e^{i\theta}-e^{-i\theta}).$$

Since the lemma of section 4 remains valid for complex-valued functions of a real variable, (18a) implies

$$e^{i(\theta_1+\theta_2)} = e^{i\theta_1}\cdot e^{i\theta_2}, \tag{19a'}$$

which reduces to the addition formula (19a) on separating the real and imaginary parts. Moreover, (9a) can now be derived by solving (17a) for t.

Finally, the significance of the isolated definition (15) of π lies in its connection with the Fourier inversion formula:

$$F(u) = \int_{-\infty}^{\infty} e^{iux}f(x)dx,$$

$$f(x) = \frac{1}{C}\int_{-\infty}^{\infty} e^{-iux}F(u)du.$$

[The first equation defines $F(u)$ as a continuous function of u if $f\in L'(-\infty,\infty)$, in particular. The additional assumption that $F\in L'(-\infty,\infty)$ implies that the second equation holds at all points of continuity of $f(x)$.] In the classical theory the constant C is defined by the integral

$$C = 2\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx.$$

In the modern theory C appears as a normalizing factor for the dual group measure on the (self-dual) additive group of the reals.* In either case the problem is to prove that $C = 2\pi$ by real variable arguments.

Set $f(x) = e^{-|x|}$. Equation (17a) or elementary integrations by parts yield

$$F(u) = 2\int_0^{\infty} e^{-x} \cos ux\, dx = \frac{2}{1+u^2}.$$

Then

$$1 = f(0) = \frac{1}{C}\int_{-\infty}^{\infty} F(u)\, du = \frac{2}{C}\int_{-\infty}^{\infty} \frac{du}{1+u^2}.$$

Hence

$$C = 2\int_{-\infty}^{\infty} \frac{du}{1+u^2} = 2\pi$$

by (15).

* For an introduction to the theory of harmonic analysis on commutative groups *cf.* the author's article Spectral theory and harmonic analysis, Proceedings of the Symposium on Spectral Theory and Differential Problems, Stillwater, Oklahoma, 1951, pp. 209–219.

THE EXPECTED NUMBER OF COMPONENTS UNDER A RANDOM MAPPING FUNCTION

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Let S be a finite set of N elements and $f(x)$ a function on S into S. A subset T of S is called *invariant* if $f^{-1}(T) = T$, or equivalently if $f(T) \subset T$ and $f^{-1}(T) \subset T$. It is straightforward to prove that if T and U are invariant, then so are the union, the intersection, and the complements of T and U. It follows immediately that the minimal non-null invariant subsets of S, called the *components of S under $f(x)$*, form a decomposition of S, *i. e.*, that every element of S belongs to exactly one component.

If $f(x)$ is one-to-one and thus a permutation of S, then the components are just the cycles of the permutation. In the general case, a component consists of a cycle together with a number of trees attached to the elements of the cycle.

Now let $f(x)$ be a random function whose values (for different arguments) are independently distributed with uniform probability $1/N$ over S. (In other

words, $f(x)$ has probability N^{-N} of being any specific mapping function.) N. Metropolis and S. Ulam* raise the problem of determining the expected number of components of S. This problem is solved here.

For any function $f(x)$ we may determine the components of S systematically in the following way. Starting with an arbitrary element x_1, if $f(x_1) \neq x_1$ we choose $x_2 = f(x_1)$, if $f(x_2) \neq x_1$ and $f(x_2) \neq x_2$ we choose $x_3 = f(x_2)$, and so on until we come to the first element x_j whose image $f(x_j)$ is one of the j elements already defined. We then call these elements the *first partial component*, and start over, beginning with an arbitrary element x_{j+1} different from the j distinct elements chosen so far. The sequence starting with x_{j+1} is continued until we reach the first element whose image is already either in the sequence or in the first partial component. If this image is in the sequence then we form the *second partial component* out of the sequence elements and start over. If it is in the first partial component, however, then we enlarge the first partial component by the addition of the sequence elements, and again start over. In either case, another arbitrary starting element generates a sequence which grows either until it runs into itself, thereby forming a new *partial component*, or else until it runs into an already existent partial component, in which case it is added on to that partial component. This process is continued until the set S is exhausted. The components of S are then just the partial components at the end.

We now apply this process to the random function $f(x)$ described above. Let $p_n(s)$ denote the probability that the growing sequence has exactly s elements just after the nth element x_n has been chosen from S. Now x_1 necessarily constitutes a sequence of exactly one element, so we have the initial condition

$$p_1(1) = 1, \qquad p_1(s) = 0 \qquad \text{for } 2 \leqq s \leqq N. \tag{1}$$

For $n > 1$, x_n can constitute a sequence of one element only if the sequence at the previous step went into a partial component (whether new or old), *i. e.*, only if $f(x_{n-1})$ was one of the $n-1$ elements $x_1, x_2, \cdots, x_{n-1}$. We have therefore the boundary condition

$$p_n(1) = \frac{n-1}{N} \qquad \text{for } 2 \leqq n \leqq N. \tag{2}$$

Moreover, x_n can be the sth element $(s > 1)$ of a growing sequence only if at the previous step the sequence had $s-1$ elements and $f(x_{n-1})$ was one of the $N-n+1$ elements not chosen by then, so we have the recursion relation

$$p_n(s) = \frac{N-n+1}{N} p_{n-1}(s-1) \qquad \text{for } 2 \leqq n \leqq N,\ 2 \leqq s \leqq N. \tag{3}$$

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* A property of randomness of an arithmetical function, this MONTHLY, vol. 60, p. 252. They call trees what are here called components. Our usage of the words "tree" and "component" is consistent with that of graph theory, if a graph is formed by considering the elements of S as points and drawing an edge from every element x to its image $f(x)$.

Let P_n be the probability of forming a new partial component during the step of going to the image of x_n. If there are s elements in the sequence at that point in the process, the probability of forming a new partial component is the probability s/N that the image will be in the sequence, so that

$$P_n = \sum_{s=1}^{N} \frac{s}{N} p_n(s). \tag{4}$$

Using formulas (2), (3), and (4) and the obvious fact that

$$\sum_{s=1}^{N} p_n(s) = 1 \qquad \text{for } 1 \leqq n \leqq N, \tag{5}$$

we proceed to derive a recursion relation for P_n. For $2 \leqq n \leqq N$ we have $p_{n-1}(N) = 0$ and hence

$$\begin{aligned} P_n &= \frac{1}{N} \cdot \frac{n-1}{N} + \sum_{s=2}^{N} \frac{s}{N} \cdot \frac{N-n+1}{N} p_{n-1}(s-1) \\ &= \frac{n-1}{N^2} + \frac{N-n+1}{N} \sum_{s=1}^{N} \frac{s+1}{N} p_{n-1}(s) \\ &= \frac{n-1}{N^2} + \frac{N-n+1}{N} \left(P_{n-1} + \frac{1}{N} \right) \\ &= \frac{N-n+1}{N} P_{n-1} + \frac{1}{N}. \end{aligned} \tag{6}$$

We wish to find E, the expected number of components. Since partial components once formed remain distinct, clearly

$$E = \sum_{n=1}^{N} P_n. \tag{7}$$

From recursion relation (6) we could already write E as an explicit sum. (Or just from formula (4) we could have written E as a double sum, since $p_n(s)$ can easily be found explicitly from formulas (1), (2), and (3).) We proceed differently, however, and find a simpler summation formula and on the way an integral representation for E which is useful in determining its asymptotic behavior.

We introduce the generating function

$$g(y) = \sum_{n=1}^{N} P_n y^n \tag{8}$$

and seek to convert the recursion relation (6) into a condition on $g(y)$. From formulas (1) and (4) we have $P_1 = 1/N$, and for convenience we define

$$P_{N+1} = \frac{1}{\mathrm{V}}, \tag{9}$$

consistently with formula (6). Multiplying formula (6) through by y^n and summing over n from 2 to $N+1$ gives

$$g(y) - \frac{y}{N} + \frac{y^{N+1}}{N} = \sum_{n=1}^{N} \frac{N-n}{N} P_n y^{n+1} + \frac{y^2 - y^{N+2}}{N(1-y)} \tag{10}$$
$$= yg(y) - \frac{y^2}{N} g'(y) + \frac{y^2 - y^{N+2}}{N(1-y)},$$

or

$$g'(y) + N \frac{1-y}{y^2} g(y) = \frac{1-y^N}{y(1-y)}. \tag{11}$$

The solution of differential equation (11) is

$$g(y) = y^N e^{N/y} \int_0^y u^{-N} e^{-N/u} \frac{1-u^N}{u(1-u)} du, \tag{12}$$

the constant of integration being determined by the condition that $g(y)$ be regular at $y=0$. Hence we have for the expected number of components

$$E = g(1) = \int_0^1 e^{N-N/u} \frac{u^{-N}-1}{u(1-u)} du, \tag{13}$$

which, making the transformation $u=N/(N+z)$, may be more simply written

$$E = \int_0^\infty \left[\left(1+\frac{z}{N}\right)^N - 1\right] e^{-z} \frac{dz}{z}. \tag{14}$$

Expanding $(1+z/N)^N$ by the binomial theorem and noting that

$$\int_0^\infty z^m e^{-z} dz = m!, \tag{15}$$

we immediately obtain

$$E = \sum_{m=1}^{N} \frac{N!}{(N-m)!\, m N^m}. \tag{16}$$

Returning to formula (14), we observe that for each value of z, as N approaches infinity $(1+z/N)^N$ approaches e^z monotonically from below. Since the integral of the limiting integrand $(1-e^{-z})/z$ diverges at infinity, it is clear that E approaches infinity monotonically with N, as was to be expected. To determine the asymptotic behavior of E for large N, we define A by the condition

$$E = \int_0^A (1-e^{-z}) \frac{dz}{z}. \tag{17}$$

Since the asymptotic behavior of E for large A is easy to determine, we seek the asymptotic behavior of A for large N. From formulas (14) and (17) we have

$$\int_0^A \left[1 - \left(1 + \frac{z}{N}\right)^N e^{-z}\right] \frac{dz}{z} = \int_A^\infty \left[\left(1 + \frac{z}{N}\right)^N - 1\right] e^{-z} \frac{dz}{z}, \tag{18}$$

or, setting $z = Aw$,

$$\int_0^1 \left[1 - e^{-N\{Aw/N - \log(1+Aw/N)\}}\right] \frac{dw}{w} = \int_1^\infty \left[e^{-N\{Aw/N - \log(1+Aw/N)\}} - e^{-Aw}\right] \frac{dw}{w}. \tag{19}$$

Now, A/N approaches zero as N approaches infinity, since otherwise the exponential in the left-hand integral of equation (19) would approach zero uniformly over every interval $0 < a \leqq w \leqq 1$ and therefore the left-hand integral itself would become infinite, whereas the right-hand integral would manifestly approach zero. Hence for any value of w

$$N\left\{\frac{Aw}{N} - \log\left(1 + \frac{Aw}{N}\right)\right\} = \frac{A^2w^2}{2N} - \frac{A^3w^3}{3N^2} + \cdots. \tag{20}$$

Accordingly, let*

$$L = \lim_{N\to\infty} \frac{A^2}{2N}. \tag{21}$$

Going to the limit under the integral signs in equation (19), which is easily justified, we obtain

$$\int_0^1 \left[1 - e^{-Lw^2}\right] \frac{dw}{w} = \int_1^\infty e^{-Lw^2} \frac{dw}{w}. \tag{22}$$

Transforming to the new variable of integration $v = Lw^2$ gives

$$\int_0^L \left[1 - e^{-v}\right] \frac{dv}{v} = \int_L^\infty e^{-v} \frac{dv}{v}, \tag{23}$$

whence

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* Note that we are not really presupposing the convergence of $A^2/2N$, since the following argument applies to the limits superior and inferior of $A^2/2N$ (proving them equal) and, properly interpreted, to a supposed infinite value of L.

$$\log L = \int_1^L \frac{dv}{v} = \int_1^\infty e^{-v} \frac{dv}{v} - \int_0^1 [1 - e^{-v}] \frac{dv}{v} \tag{24}$$
$$= \int_0^\infty e^{-v} \log v \, dv,$$

using integration by parts. But this last integral is known* to equal $-C$, where $C = 0.5772 \cdots$ is Euler's constant. For large N, therefore,

$$\log A = \tfrac{1}{2}(\log 2N - C) + o(1), \tag{25}$$

where $o(1)$ denotes a term vanishing in the limit. It would not be difficult to obtain further terms of the asymptotic series for $\log A$.

Returning to formula (17), it is clear that E behaves like $\log A$ for large A; indeed,

$$E - \log A = \int_0^1 (1 - e^{-z}) \frac{dz}{z} - \int_1^A e^{-z} \frac{dz}{z} \tag{26}$$
$$= C + \int_A^\infty e^{-z} \frac{dz}{z} = C + o(1).$$

The desired asymptotic behavior of the expected number E of components for large N is therefore

$$E = \tfrac{1}{2}(\log 2N + C) + o(1). \tag{27}$$

Formulas (16) and (27) may be compared to the corresponding results when $f(x)$ is a random permutation. In that case the expected number E of components (cycles) is given by†

$$E = \sum_{m=1}^{N} \frac{1}{m} = \log N + C + o(1). \tag{28}$$

* See for instance D. Bierens de Haan, Nouvelles tables d'intégrales définies, Leyden, 1867, table 256, formula (1).

† See for instance R. E. Greenwood, The number of cycles associated with the elements of a permutation group, this MONTHLY, vol. 60, p. 407.

THE RELATIONSHIP BETWEEN HARMONIC AND ANHARMONIC COLLINEATIONS IN A PLANE

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Introduction. A plane collineation (Ω) is said to be "regular" when there exist at least three *non-collinear* invariant points A, B, C, forming a triangle (Δ), such that Ω can be exhibited in the analytic form:

(I) $$\rho x' = \lambda x, \qquad \rho y' = \mu y, \qquad \rho z' = \nu z,$$

(ρ being a factor of proportionality), where (x, y, z) and (x', y', z') are respectively the triads of homogeneous or projective coordinates of a pair of corresponding points P, Q (referred to Δ as the *fundamental* triangle), and λ, μ, ν are certain constants, called the "roots" of Ω. In particular, Ω is called "anharmonic,"* when one of the three parameters λ, μ, ν is the geometric mean between the other two. More particularly, the "anharmonic" collineation is called "harmonic,"† when the "roots" are given by:

$$\lambda:\mu:\nu = -1:1:1 \quad \text{or} \quad 1:-1:1 \quad \text{or} \quad 1:1:-1.$$

The main purpose of the present paper is to reckon with the relationship between an "A.C." and an "H.C." The precise result to be proved is that the product of *two* H.C.'s is an A.C. and conversely an A.C. is always expressible—and that in an infinity of ways—as the product of two H.C.'s.

1. Starting with *two* given H.C.'s S_1, S_2, *viz.*,

$$S_1 \equiv \Pi_{O_1, L_1} \quad \text{and} \quad S_2 \equiv \Pi_{O_2, L_2},$$

let us now consider the two correlated product-collineations Π, Π', where

$$\Pi \equiv S_1 S_2 \quad \text{and} \quad \Pi' \equiv S_2 S_1.$$

Palpably the point of intersection (A) of the two base-lines L_1, L_2 (as shown in the adjoining figure) is an *invariant* point for both the collineations Π and Π'. To find the other two *invariant* points for either of them, we firstly note the two points K_1, K_2, where the lines L_1, L_2 cut the line $O_1 O_2$, and secondly mark the two *foci* (or *focal points*) B, C of the involution, defined uniquely by the point-

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* If ξ, η, ζ denote respectively the cross-ratios of the *three* pencils of lines, *viz.*, $(b, c; AQ, AP)$, $(c, a; BQ, BP)$ and $(a, b; CQ, CP)$, where P, Q are as before a pair of corresponding points and a, b, c are the *three* sides of Δ, an "anharmonic" collineation,—often contracted as an A.C.—is definable alternatively as one for which two out of the three associated entities (ξ, η, ζ) are *equal*. The trivial case $(\xi = \eta = \zeta)$, answering to an "identity," may be left out of consideration.

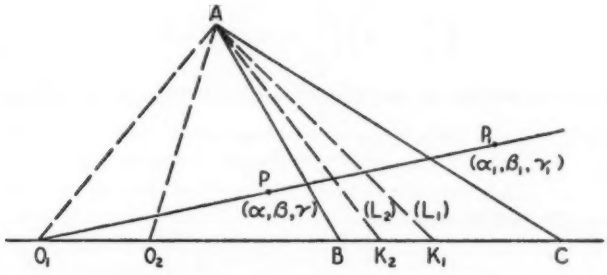
† A "harmonic" collineation—often contracted as an H.C.—is best defined as a particular variety of (1, 1)-correspondence between pairs of points P, Q, which are so related that when any one of these points (say, P) is joined to a certain *fixed* point (O) and the line OP cuts a certain *fixed* line (L) (not passing through O) at a point R, then Q must lie on the line OPR and the pairs O, R and P, Q must be harmonic conjugates. The H.C. is said to have O for its "centre" and L for its "axis" and is symbolised as $\Pi_{O,L}$, and can be put in the analytic form:

$$(\Pi_{O,L}): \quad \rho x' = -x, \ \rho y' = y, \ \rho z' = z,$$

provided that O is chosen as the *first* angular point (A) and any two distinct points (B, C) on L are chosen as the *second* and *third* angular points of the "triangle of reference" (Δ). For a *different ordering* of the three angular points of Δ, the H.C. can be made to assume either of the two forms:

$$(\rho x' = x, \ \rho y' = -y, \ \rho z' = z),$$
$$(\rho x' = x, \ \rho y' = y, \ \rho z' = -z).$$

pairs (O_1, K_1) and (O_2, K_2). Then the points B, C, being harmonically conjugate with respect to both the point-pairs (O_1, K_1) and (O_2, K_2), must simply interchange their positions under the operation of either S_1 or S_2. So no matter whether the operation of S_1 is preceded by or followed by that of S_2, each of the two points B, C will, after suffering a double interchange, remain absolutely *stationary*. Thus A, B, C are the three common invariant points of both the product collineations Π, Π'.



Now let us take the resulting invariant triangle ABC for the "triangle of reference." Then since the individual equations of AB, AC are $z=0$ and $y=0$, and the line-pair (AB, AC) is harmonic with each of the line-pairs (AO_1, AK_1) and (AO_2, AK_2), the equations of these four lines AO_1, AK_1, AO_2, AK_2 may respectively be taken as:

$$y = mz, \qquad y = -mz, \qquad y = nz, \qquad y = -nz,$$

where m and n are certain constants. Evidently the two points O_1, O_2 are $(0, m, 1)$ and $(0, n, 1)$.

If, then, an arbitrary point $P(\alpha, \beta, \gamma)$ be changed into $P_1(\alpha_1, \beta_1, \gamma_1)$ by the operation of S_1, the combined equation to the line-pair (AP, AP_1) is:

$$y^2 - \left(\frac{\beta}{\gamma} + \frac{\beta_1}{\gamma_1}\right) yz + \frac{\beta\beta_1}{\gamma\gamma_1} z^2 = 0. \tag{1}$$

Since the combined equation to the line-pair (AO_1, AK_1) is

$$y^2 - m^2 z^2 = 0, \tag{2}$$

and (1) and (2) are harmonically conjugate, we must have

$$\frac{\beta_1}{\gamma_1} = m^2 \frac{\gamma}{\beta}. \tag{3}$$

Further the collinearity of the three points O_1, P and P_1 imposes the condition:

$$(4)\qquad \begin{vmatrix} \dfrac{\alpha_1}{\gamma_1}, & \dfrac{\beta_1}{\gamma_1}, & 1 \\ \dfrac{\alpha}{\gamma} & \dfrac{\beta}{\gamma}, & 1 \\ 0, & m, & 1 \end{vmatrix} = 0.$$

When (3) is substituted in (4), the latter equation takes the form:

$$(5)\qquad \left(\frac{\beta}{\gamma} - m\right)\left(\frac{\alpha_1}{\gamma_1} + m\frac{\alpha}{\beta}\right) = 0.$$

Since $(\beta/\gamma) - m \neq 0$ for an arbitrary position of $P(\alpha, \beta, \gamma)$, (5) simplifies to:

$$(6)\qquad \frac{\alpha_1}{\gamma_1} = -m\frac{\alpha}{\beta}.$$

So combining (3) and (6), we get

$$\alpha_1 : \beta_1 : \gamma_1 = -m\alpha : m^2\gamma : \beta.$$

If we now represent the coordinates of the pair of corresponding points P, P_1 by the more familiar notations (x, y, z) and (x_1, y_1, z_1), then the H.C. (S_1) can be put in the analytical form:

$$(7)\qquad S_1: \quad \rho_1 x_1 = -mx, \quad \rho_1 y_1 = m^2 z, \quad \rho_1 z_1 = y,$$

where ρ_1 is a factor of proportionality.

From symmetry S_2 can be written as:

$$(8)\qquad S_2: \quad \rho_2 x_2 = -nx, \quad \rho_2 y_2 = n^2 z, \quad \rho_2 z_2 = y,$$

where ρ_2 has a similar meaning, and (x, y, z), and (x_2, y_2, z_2) are the initial and final positions of an *arbitrary* point.

By a simple algebraic superposition of the two transformations, we can easily obtain the product Π in the form:

$$(9)\qquad \Pi = S_1 S_2: \quad \rho x' = mnx, \quad \rho y' = n^2 y, \quad \rho z' = m^2 z.$$

By a simple interchange of the two parameters m, n, we find Π' in the form:

$$(10)\qquad \Pi' \equiv S_2 S_1: \quad \rho x' = nmx, \quad \rho y' = m^2 y, \quad \rho z' = n^2 z.$$

Now each of the two sets of equations (9) and (10) being of the *symbolic* form:

$$(11)\qquad \rho x' = \lambda x, \quad \rho y' = \mu y, \quad \rho z' = \nu z,$$

where

$$\lambda^2 = (mn)^2 = m^2 n^2 = \mu\nu,$$

it follows that each of the two products Π, Π' is an A.C. This disposes of the first part of the premised proposition *viz.*, that *the product of two H.C.'s is an A.C.*

In order to test whether the commutative law holds or not we have simply to ascertain the condition under which the two sets of equations (9) and (10), defining (Π) and (Π'), may be *identical*. The requisite conditions are easily seen to be:

$$\frac{mn}{nm} = \frac{n^2}{m^2} = \frac{m^2}{n^2},$$

which imply that

$$m = +n \quad \text{or} \quad m = -n.$$

The first contingency is untenable, for the equality $m = +n$ implies that the points O_1, K_1 (Fig.) coincide *respectively* with O_2, K_2, leading ultimately to the coincidence of S_1 and S_2. Next the other contingency (*viz.*: $m = -n$) is possible only when the points O_1, K_1 coincide *respectively* with K_2, O_2.

Summing up the results, we can now assert that *the product of two H.C.'s conforms to the commutative law, when and only when the "centre" of either of them lies on the "axis" of the other.*

2. To examine the converse proposition, we observe that an arbitrary A.C. (T), given initially in the canonical form (11)—with *known* constants λ, μ, ν compatible with the relation $\lambda^2 = \mu\nu$—can be alternatively exhibited in the form (10), provided that the constants m, n are properly chosen. That is to say, *any* A.C. (T) can be expressed in the product form S_2S_1, where S_1, S_2 are the two H.C.'s given by (7) and (8).

Next to show that there are *infinitely many* solutions, we may modify S_1, S_2 by introducing an *arbitrary* parameter k as under:

$$S_1: \quad \rho x' = -mx, \quad \rho y' = km^2 z, \quad \rho z' = \frac{y}{k},$$

$$S_2: \quad \rho x' = -nx, \quad \rho y' = kn^2 z, \quad \rho z' = \frac{y}{k}.$$

These are all reflexions (H.C.'s), which lead to the *same* product, whatever be the value of the parameter k.

Recapitulating the results, we finally affirm that the product of two H.C.'s is an A.C. and that conversely every A.C. can be factorised into two H.C.'s, and that in an infinity of ways.

In conclusion we beg to express our indebtedness to the referee for his suggestions and helpful criticisms.

MATHEMATICAL NOTES

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ON THE POLYNOMIAL SOLUTIONS OF A RICCATI EQUATION

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1. The problem. In this note we propose to find the polynomial solutions of

$$Ay' = B_0 + B_1 y + B_2 y^2 \tag{1}$$

where A, B_0, B_1, B_2 are polynomials in x of degrees a, b_0, b_1, b_2, respectively. A general method is developed for finding all polynomial solutions, and some modifications suitable for special cases are given.

2. The degrees of polynomial solutions. Let m be the degree of a polynomial solution of (1). In a previous note [1] it was shown that if $a \neq 1+b_1$ then m is one of the five numbers $1+b_0-a$, $a-b_2-1$, b_0-b_1, $(b_0-b_2)/2$, b_1-b_2; if $a=1+b_1$ then $b_0-b_1 \leqq m \leqq b_1-b_2$. This result can be sharpened. Let

$$y = cx^m + c_1x^{m-1} + \cdots + c_m, \qquad c \neq 0,$$

and let α, β_0, β_1, β_2 be the leading coefficients of A, B_0, B_1, B_2 respectively. For obvious reasons we assume $\alpha \neq 0$, $\beta_2 \neq 0$. Writing only the leading powers of each term in (1) we have

$$mc\alpha x^{a+m-1} + \cdots = \beta_0 x^{b_0} + \beta_1 c x^{b_1+m} + \beta_2 c^2 x^{b_2+2m} + \cdots . \tag{2}$$

Assume $a-1 \neq b_1$. If $m=1+b_0-a$ then b_0 is the largest exponent in (2). A value of m smaller than $1+b_0-a$ is not possible in this case since (2) would contain no term to balance $\beta_0 x^{b_0}$; the only possible m larger than $1+b_0-a$ makes $a+m-1=b_2+2m$, *i.e.*, $m=a-b_2-1$. In the same way it is seen that if $m=b_0-b_1$ then the only other possible value of m is b_1-b_2. If $a-1=b_1$ then the two cases considered coincide and nothing is changed unless, at the same time,

$$a-1=b_1,\ \frac{\beta_1}{\alpha} \text{ is a positive integer, } b_0 < b_1 + \frac{\beta_1}{\alpha},\ b_2 < b_1 - \frac{\beta_1}{\alpha}. \tag{3}$$

In this case, as is seen from (2), $m=\beta_1/\alpha$ is another possible degree for a polynomial solution. Finally, if none of the mentioned cases holds, but $m=(b_0-b_2)/2$, then no other degree is possible. Thus, we have

THEOREM 1. *The polynomial solutions of* (1) *have degrees from exactly one of the following classes*:

(i) $(b_0-b_2)/2$,

(ii) $1+b_0-a$, $a-b_2-1$,

(iii) b_0-b_1, b_1-b_2,

(iv) $1+b_0-a=b_0-b_1$, $a-b_2-1=b_1-b_2$, β_1/α.

Class (iv) *can occur only if conditions* (3) *are satisfied.*

If conditions (3) are satisfied then $m=\beta_1/\alpha$ is referred to as the "singular exponent."

It is seen that (1) can never have polynomial solutions of more than three different degrees. That three different degrees can occur is shown by the example given in [1].

3. The algorithm for the polynomials. The coefficients c, c_1, $\cdots$, c_m of the polynomial solution y can now be determined by the following method. If m is not the singular exponent, then equating the coefficient of the highest power of x in (2) to zero gives a linear or quadratic equation for c, hence at most two possible values for c. To determine the other coefficients of y put $y=cx^m+y_1$ in (1) and find the following equation for y_1:

$$Ay_1' = (B_0 + B_1cx^m - Acmx^{m-1} + B_2c^2x^{2m}) + (B_1 + 2B_2cx^m)y_1 + B_2y_1^2.$$

This is another Riccati equation for y_1 with known polynomial coefficients. By proceeding as before we determine the leading coefficient c_1 of y_1, and c_2, $\cdots$, c_m follow in similar fashion.

It remains to consider the case in which m is the singular exponent β_1/α. Equating the coefficient of the highest power of x in (2) to zero then gives $m\alpha c=\beta_1 c$; hence c remains undetermined in this first step. Comparing the coefficients of the powers x^{a+m-2} gives an equation

$$(m-1)\alpha c_1 + \cdots = \beta_1 c_1 + \cdots,$$

where the terms not written out contain c and possibly c^2, but not c_1. Since $(m-1)\alpha-\beta_1=-\alpha\neq 0$, c_1 is uniquely determined as a function of c, c^2. By proceeding in analogous fashion, we determine c_2 uniquely as a function of c, c^2, c^3, c^4, $\cdots$, and c_m uniquely as a function of c, c^2, $\cdots$, c^{2m} by comparing the coefficients of the powers $x^{a-1}=x^{b_1}$ in (2). Finally, comparing the coefficients of x^{b_1-1}, $\cdots$, x^0 gives one or more equations for c alone, from which the possible values of c are determined. These, in turn, give the possible values of c_1, $\cdots$, c_m.

If it is known that (1) has more than one polynomial solution, the following algorithm gives all solutions. The difference $u=y-y_0$ between two polynomial solutions of (1) is a polynomial solution of the simplified Riccati equation

$$Au' = (B_1 + 2y_0B_2)u + B_2u^2. \tag{1'}$$

Suppose r (real or complex) is a zero of $u(x)$ of multiplicity p. Then r is a zero of the right-hand term of (1') of multiplicity $\geqq p$, and since r is a zero of $u'(x)$ of multiplicity $p-1$, r must be a zero of $A(x)$. It follows that the polynomial solutions of (1') are of the form

$$u = k(x-r_1)^{p_1}(x-r_2)^{p_2}\cdots(x-r_h)^{p_h}, \tag{4}$$

where the r_i are the zeros of $A(x)$, the p_i are non-negative integers, and k is a constant. Since for a given u, y_0 is immediately determined from (1'), we have

Theorem 2. *If* (1) *has more than one polynomial solution then the class of functions*

$$y = (Au' - B_1u - B_2u^2)/2B_2u, \tag{5}$$

where u *is given by* (4), *includes all polynomial solutions of* (1).

This algorithm is not applicable if (1) has only one polynomial solution. In this case (1') has only the trivial solution $u=0$ and y_0 cannot be determined from (1').

Finally we remark that if k distinct zeros $r_1, \cdots, r_k$ of $A(x)$ are known, then any polynomial solution of (1) of degree $\leqq k-1$ is readily determined. If $A(r_i)=B_1(r_i)=B_2(r_i)=0$ then it follows from (1) that no polynomial solution can exist unless also $B_0(r_i)=0$. In the latter case $x-r_i$ is a common factor of A, B_0, B_1, B_2, and may be eliminated from (1). Hence, we may assume that not both $B_1(r_i)$, $B_2(r_i)$ vanish. Substitution of $x=r_i$ in (1) then gives a linear or quadratic equation for $y(r_i)$, hence at most two values for $y(r_i)$. When $y(r_1), \cdots, y(r_k)$ are known, the polynomial $y(x)$ of degree $\leqq k-1$ is found by interpolation.

4. An example. The Riccati equation

$$(x - x^3)y' = 4x - 4x^2 - (x^2 - 4x + 1)y - y^2 \tag{6}$$

has the four polynomial solutions: $2x$, $x-1$, $3x-1$, x^2+2x-1. By Theorem 1, the possible degrees are 0, 1, 2. The possible polynomials (4) are:

$$k_0, \quad k_1x, \quad k_2(1+x), \quad k_3(1-x), \quad k_4x(1+x), \quad k_5x(1-x),$$
$$k_6(1-x^2), \quad k_7x^2, \quad k_8(1+x)^2, \quad k_9(1-x)^2.$$

Putting $u=k_0$ in (5) we find the polynomial $-(x^2-4x+1+k_0)/2$ and direct substitution in (6) shows that this is not a solution for any k_0. Next put $u=k_1x$ in (5) and obtain the polynomial $\frac{1}{2}(4-k_1)x-1$, which on substitution in (6) gives $k_1=\pm 2$. Both $x-1$ and $3x-1$ are polynomial solutions. These solutions are obtained again if $k_2(1+x)$ and $k_3(1-x)$ are used for u in (5). Putting $u=k_4x(1+x)$ we find $k_4=-1$ and the solution x^2+2x-1. Putting $u=k_6(1-x^2)$ we find $k_6=-1$ and the solution $2x$. The remaining possible u's yield no further polynomial solutions of (6).

These solutions could also have been found from the values $y(0)=0$ or -1, $y(1)=1$ or 2, $y(-1)=-2$ or -4, which result from the substitution of $x=0, 1, -1$ in (6).

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A SET-THEORETIC DESCRIPTION OF NORMAL TOPOLOGIES*

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1. Introduction. A matter of considerable interest in topology is the application of the topology of a space to obtain auxiliary structures by means of which the space may be studied algebraically. This general approach includes such diverse matters as homology and cohomology theories [2], homotopy theory [3], lattice-theoretic methods such as those of Stone [7] and Wallman [10], and algebraic characterizations of spaces by functions on them. Perhaps the most elegant example of this last type is the theory of Kaplansky [4] which characterizes compact Hausdorff spaces in terms of lattices of continuous functions. It is the object of the present note to characterize convergence in normal spaces algebraically (set-theoretically).

By a space we shall mean a set with certain distinguished subsets called open so that:

(i) The null set is open.

(ii) Any set-theoretic sum of open sets is open.

(iii) Any finite set-theoretic product of open sets is open.

(iv) The complement of any finite set is open. In the presence of (iv), (ii) implies

(i′) The entire space is open.

A set is called closed if and only if its complement is open. The complement of a set F is written $C(F)$. The smallest closed set which contains a set F is called the closure of F and written $\overline{F}$.

By a net [5, 6], we mean any mapping of a directed set [1, 5, 6, 8] into a set. Let $x(n):D\to S$ be a net mapping a directed set D into a set S. It is conventional to write x_n for $x(n)$. If S is a space, one says that x_n converges to $x\in S$, provided for each open set G in S with $x\in G$ there is $a\in D$ so that $b\geqq a$ implies $x_b\in G$. Kelley [5; page 9, no. 7] has shown that x is an accumulation point of a set $E\subset S$ if and only if there is a net taking values in $E-(x)$ and converging to x. Thus we have

LEMMA 1. *The topology of a space is completely determined by the convergence of nets in the space.*

By Lemma 1, then, if we find a structure which specifies the convergence of nets in a space this structure will be an adequate description of the topology of the space.

If A_n is a net on D whose values are subsets of a set S, we define

$$\liminf_n A_n = \bigcup_{a\in D} \bigcap_{b\geqq a} A_b \quad [1; \text{page 40, no. 2; page 60, no. 7}].$$

This, we note, is a purely algebraic (or, if one prefers, set-theoretic) construction without reference to any topology on S.

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2. The mapping $\mathfrak{k}: S \to \mathfrak{R}$. Let S be a space. If A and B are non-null subsets of S, a characteristic function for A and B is a continuous mapping $f: S \to [0, 1]$ of S into the closed unit interval so that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$. Clearly, if $f: S \to [0, 1]$ is a characteristic function for A and B, then $g(x) = 1 - f(x)$ is a characteristic function for B and A. We denote by K the class of all functions $f: S \to [0, 1]$ with the property: There are two non-null closed subsets A and B of S so that at least one of A and B contains a non-null open set and $f: S \to [0, 1]$ is a characteristic function for A and B. We denote by $\mathfrak{R}$ the Boolean algebra of subsets of K [1; page 153, paragraph 2] and define a mapping $\mathfrak{k}: S \to \mathfrak{R}$ by taking as $\mathfrak{k}(x)$ all those members of K which vanish on some open set containing x. We note that if $x_n: D \to S$ is a net in S, then $\mathfrak{k}(x_n): D \to \mathfrak{R}$ is a net in $\mathfrak{R}$.

LEMMA 2. *If x_n is a net in S which converges to $x \in S$, then* $\liminf_n \mathfrak{k}(x_n) \supset \mathfrak{k}(x)$.

Proof. Suppose $f \in \mathfrak{k}(x)$. Then f vanishes on some open set U with $x \in U$. But since x_n converges to x, there is $a \in D$ so that $b \geqq a$ implies $x_b \in U$. Hence,

$$f \in \bigcap_{b \geqq a} \mathfrak{k}(x_b) \subset \liminf_n \mathfrak{k}(x_n).$$

3. Additional terminology. Urysohn's Lemma. A space S is called completely regular [9; page 149, no. 42.1] provided there is a characteristic function for each pair A and B where A consists of a single point and B is a non-null closed set which does not contain the point of A. We shall write merely x for the set having only x as a member. S is called normal [9; page 79, no. 22.5; page 94, no. 28] provided when E and F are disjoint closed sets there are disjoint open sets G and H with $E \subset G$ and $F \subset H$. Any normal space is completely regular and we shall employ the stronger result [9; page 150, no. 43.3; page 149; no. 42.3; page 94, no. 28; page 79, no. 22.9]:

LEMMA 3 (*Urysohn's Lemma*). *If S is normal and A and B are disjoint, non-null, closed subsets of S, there exists a characteristic function for A and B.*

We note also that complete regularity (and, hence, normality) of S implies that S has the property of regularity [9; page 92, no. 29; page 81, no. 22.9; page 149, no. 43.2]: Any open set containing a closed set E contains the closure of an open set which contains E.

We say that two sets A and B are incomparable if neither contains the other. A class of sets which are pairwise incomparable is called totally unordered.

4. Main theorem.

LEMMA 4. *If S is completely regular and if x and y are distinct points of S, then $\mathfrak{k}(x)$ and $\mathfrak{k}(y)$ are incomparable.*

Proof. Suppose the hypotheses. Since S is completely regular, it is regular and there is an open set G with $y \in G \subset \overline{G} \subset C(x)$. Also, since S is completely regular, there is a characteristic function f for $\overline{G}$ and x. Thus, $f \in \mathfrak{k}(y)$ but $f \notin \mathfrak{k}(x)$.

Similarly, $\mathfrak{k}(x) \not\subset \mathfrak{k}(y)$.

LEMMA 5. *If S is normal and x_n is a net in S, and if $x \in S$ with* $\liminf_n \mathfrak{k}(x_n) \supset \mathfrak{k}(x)$, *then x_n converges to x.*

Proof. Suppose the hypotheses and the contradiction of the conclusion. Then there is an open set U with $x \in U$, and $a \in D$ implies there is $b \in D$ with $b > a$ and $x_b \in C(U)$. Since S is normal, it is regular, and there is an open set G with $x \in G \subset \overline{G} \subset U$. Also, since S is normal, Lemma 3 applies and there is a characteristic function f for the pair $\overline{G}$ and $C(U)$. But $f \in \mathfrak{k}(x)$ and $f \notin \lim\inf_n \mathfrak{k}(x_n)$ since $f(x_b) = 1$ for some x_b with b exceeding any preassigned $a \in D$. Thus, Lemma 5 is proved by contradiction.

Collecting the results of Lemmas 2, 4, and 5 we have

THEOREM. *If S is a normal space, there is a biuniform mapping $\mathfrak{k}: S \to \mathfrak{R}$ into a totally unordered subset of a set algebra $\mathfrak{R}$ so that if x_n is a net in S then x_n converges to $x \in S$ if and only if* $\lim\inf_n \mathfrak{k}(x_n) \supset \mathfrak{k}(x)$.

In view of Lemma 1, this theorem accomplishes the objective of algebracizing the topology of normal spaces.

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CONGRUENCES FOR THE NUMBER OF n-GONS FORMED BY n LINES

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1. Introduction. Let g_n denote the number of polygons of n sides (including degenerate cases) formed by a network of n lines. Robinson [3] proved that g_n satisfies the recurrence

$$g_{n+1} = ng_n + \tfrac{1}{2}n(n-1)g_{n-2}, \tag{1.1}$$

when $g_1 = g_2 = 0$, $g_3 = 1$. For proofs of the relation

$$\lim_{n=\infty} \frac{u_n^2}{n} = \frac{4}{e^{3/2}\pi} \qquad (g_n = \tfrac{1}{2}(n-1)!u_n),$$

see [4].

In the present note we establish certain congruences satisfied by g_n. If m is a fixed integer greater than or equal to 1 we show first that

$$g_{n+m} \equiv g_m g_n \pmod{m_0} \qquad (n \geqq 1), \tag{1.2}$$

where $m_0 = m$ for m odd, $m_0 = \frac{1}{2}m$ for m even. Formula (1.2) implies $g_{n+mk} \equiv g_m^k g_n$ so that the period (mod m_0) is mk when $g_n^k \equiv 1$. More generally if we define

$$\Delta^r g_n = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} g_{n+sm} g_{(r-s)m}, \tag{1.3}$$

then we show that

$$\Delta^r g_n \equiv 0 \pmod{m_0^{[(r+1)/2]}} \qquad (n \geqq 1), \tag{1.4}$$

where $[x]$ denotes the greatest integer less than or equal to x. To put (1.2) in more explicit form it is necessary to know the residue of g_m (mod m). For $m = p$, a prime greater than 2, we show that

$$g_p \equiv \tfrac{1}{2}(p-1) \pmod{p}. \tag{1.5}$$

The above results are suggested by similar results satisfied by the number of three-line latin rectangles [1], [2].

2. Proof of (1.2). Replacing n by $n+m$ in (1.1) we get

$$g_{n+m+1} = (n+m)g_{n+m} + \tfrac{1}{2}(n+m)(n+m-1)g_{n+m-2},$$

so that

$$\Delta g_{n+1} = n\Delta g_n + m g_{n+m} + \tfrac{1}{2}n(n-1)\Delta g_{n-2} + \tfrac{1}{2}m(m+2n-1)g_{n-m+2}, \tag{2.1}$$

where $\Delta g_n = g_{n+m} - g_n g_m$. Clearly (2.1) implies

$$\Delta g_{n+1} \equiv n\Delta g_n + \tfrac{1}{2}n(n-1)\Delta g_{n-2} \pmod{m_0}, \tag{2.2}$$

where m_0 has the same meaning as in (1.2). Now for $n = m$, (1.1) yields

$$\Delta g_1 = g_{m+1} = m g_m + \tfrac{1}{2}m(m-1)g_{m-2} \equiv 0. \tag{2.3}$$

Similarly we have

$$\Delta g_2 = g_{m+2} = (m+1)g_{m+1} + \tfrac{1}{2}(m+1)m g_{m-1} \equiv 0, \tag{2.4}$$

$$\Delta g_3 = g_{m+1} - g_m = (m+2)g_{m+2} + \tfrac{1}{2}(m+2)(m+1)g_m - g_m \equiv 0. \tag{2.5}$$

But (2.2) together with (2.3), (2.4), (2.5) imply

$$\Delta g_n \equiv 0 \pmod{m_0} \qquad (n \geqq 1), \tag{2.6}$$

which is equivalent to (1.2).

3. Proof of (1.4). The proof of (1.4) is somewhat more elaborate. In the first place we require an extension of (2.1). Using the notation (1.3), we have for $n \geqq 3$

$$\begin{aligned}\Delta^r g_{n+1} = {} & n\Delta^r g_n + \tfrac{1}{2}n(n-1)\Delta^r g_{n-2} + rm\Delta^{r-1}g_{n+m} \\ & + \tfrac{1}{2}rm(m+2n-1)\Delta^{r-1}g_{n+m-2} + \tfrac{1}{2}r(r-1)m^2\Delta^{r-2}g_{n+2m-2},\end{aligned} \tag{3.1}$$

as is easily verified. We shall prove that

$$\Delta^{2r-1} g_n \equiv 0 \equiv \Delta^{2r} g_n \pmod{m_0^r} \qquad (n \geqq 1). \tag{3.2}$$

For $r=1$, (3.2) is a consequence of (2.6). We therefore assume that (3.2) holds up to and including the value $r-1$. Then (3.1) implies

$$\Delta^r g_{n+1} \equiv n\Delta^r g_n + \tfrac{1}{2}n(n-1)\Delta^r g_{n-2} \pmod{m_0^{[(r+1)/2]}},$$

so that it suffices to prove (3.2) for $n=1, 2, 3$.

Now for $n=1$, we have, using (1.1) and (3.1),

$$\begin{aligned}\Delta^r g_1 &= \sum_{s=1}^{r} (-1)^{r-s}\binom{r}{s}(smg_{sm} + \tfrac{1}{2}sm(sm-1)g_{sm-2})g_{(r-s)m} \\ &= mr\sum_{1}^{r}(-1)^{r-s}\binom{r-1}{s-1}g_{sm}g_{(r-s)m} \\ &\quad + \tfrac{1}{2}m^2r(r-1)\sum_{2}^{r}(-1)^{r-s}\binom{r-2}{s-2}g_{sm-2}g_{(r-s)m} \\ &\quad + \tfrac{1}{2}m(m-1)r\sum_{1}^{r}(-1)^{r-s}\binom{r-1}{s-1}g_{sm-2}g_{(r-s)m} \\ &= mr\Delta^{r-1}g_m + \tfrac{1}{2}m^2r(r-1)\Delta^{r-2}g_{2m-2} + \tfrac{1}{2}m(m-1)r\Delta^{r-1}g_{m-2}.\end{aligned}$$

Hence by the inductive hypothesis we infer that (3.2) holds for $n=1$. In exactly the same way

$$\Delta^r g_2 = \Delta^r g_1 + mr\Delta^{r-1}g_{m+1} + \tfrac{1}{2}m^2r(r-1)\Delta^{r-2}g_{2m-1} + \tfrac{1}{2}m(m-1)r\Delta^{r-1}g_{m-1},$$

and therefore (3.2) holds for $n=2$.

As for $n=3$, we have

$$\begin{aligned}\Delta^r g_3 = {} & 2\Delta^r g_2 + mr\Delta^{r-1}g_{m+2} + \tfrac{1}{2}m^2r(r-1)\Delta^{r-2}g_{2m} \\ & + \tfrac{1}{2}m(m+3)r\Delta^{r-1}g_m + \Delta^r g_0,\end{aligned} \tag{3.3}$$

where we have put

$$\Delta^r g_0 = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} g_{sm} g_{(r-s)m}, \qquad (g_0 = 1). \tag{3.4}$$

Now it is clear from (3.4) that

$$\Delta^{2r-1} g_0 = 0 \tag{3.5}$$

and

$$\Delta^{2r} g_0 = 2\Delta^{2r-1} g_m. \tag{3.6}$$

Hence for r odd it follows from (3.3) and (3.5) that $\Delta^r g_0 = 0$, while for r even the same result is a consequence of (3.3), (3.6) and the inductive hypothesis. This completes the proof of (3.2).

4. Proof of (1.5). By formula (2) of [4] we have

$$1 + \sum_{1}^{\infty} \frac{g_n x^n}{n!} = (1 - x)^{-1/2} \exp\left(-\tfrac{1}{4}(x^2 + 2x)\right), \tag{4.1}$$

which is indeed an easy consequence of (1.1). Now (4.1) implies the following explicit expression for g_m:

$$g_m = \sum_{r+s+2t=m} (-1)^{s+t} \frac{m!}{r!s!(2t)!} C_r C_t 2^{-s-t}, \tag{4.2}$$

where $C_r = 1 \cdot 3 \cdot 5 \cdots (2r-1)$. For $m = p$, an odd prime, it is evident that all terms in the right member of (4.2) are divisible by p except the term corresponding to $r=0$, $s=p$, $t=0$. We accordingly get

$$g_p \equiv -2^{-p} \equiv \frac{p-1}{2} \pmod{p},$$

which proves (1.5).

Combining (1.5) and (1.2) we see that

$$g_{n+p} \equiv -\tfrac{1}{2} g_n \pmod{p}, \qquad (n \geqq 1). \tag{4.3}$$

5. The following small table of the g_n and their residues was kindly supplied by John Riordan.

n	0	1	2	3	4	5	6	7	8	9	10
g_n	1	0	0	1	3	12	70	465	3507	30016	286884
mod 3	1	0	0	1	0	0	1	0	0	1	0
4	1	0	0	1	3	0	2	1	3	0	0
5	1	0	0	1	3	2	0	0	2	1	4

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ON THE EVALUATION OF DIRICHLET'S INTEGRAL

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The well-known multiple integral

$$\iint_{(R_n)} \cdots \int x_1^{\alpha_1-1} x_2^{\alpha_2-1} \cdots x_n^{\alpha_n-1}(1 - x_1 - \cdots - x_n)^{\alpha_0-1} dx_1 dx_2 \cdots dx_n,$$

where R_n is the region defined by $x_1 \geqq 0$, $x_2 \geqq 0$, $\cdots$, $x_n \geqq 0$, $x_1+x_2+\cdots+x_n \leqq 1$, and $\alpha_0, \alpha_1, \cdots, \alpha_n$ are real positive constants, is usually evaluated by the use of Dirichlet's transformation. The integral is expressed as a product of Beta functions and the procedure does not at any stage suggest the relation between the Beta and Gamma functions.

The following is an alternative procedure for evaluating the above integral by the use of Laplace Transforms. It has the advantage that when $n=1$ the procedure provides a proof of the relation between the Beta and Gamma functions.

Let $f_0(x), f_1(x), \cdots, f_n(x)$, be $n+1$ functions whose Laplace Transforms *viz.*, $\int_0^\infty e^{-px} f_r(x)dx$ $(r=0, 1, \cdots, n)$ exist. We have by the convolution theorem:

$$L(f_0(x) * f_1(x) * \cdots * f_n(x)) = Lf_0(x) \cdot Lf_1(x) \cdots Lf_n(x). \tag{1}$$

Now $f_0(x) * f_1(x) * \cdots * f_n(x)$ can be written as

$$\int_0^x f_n(x_n)dx_n [f_0(x) * f_1(x) * \cdots * f_{n-1}(x)]_{(x-x_n)} = \cdots = \cdots$$

$$= \int_0^x f_n(x_n)dx_n \int_0^{x-x_n} f_{n-1}(x_{n-1})dx_{n-1} \cdots$$

$$\int_0^{x-x_n-\cdots-x_2} f_1(x_1)f_0(x - x_n - \cdots - x_2 - x_1)dx_1.$$

The inverse of (1) is

$$f_0(x) * f_1(x) * \cdots * f_n(x) = L^{-1}\{Lf_0(x) \cdot Lf_1(x) \cdots Lf_n(x)\}.$$

Therefore

$$\int_0^x f_n(x_n)dx_n \int_0^{x-x_n} f_{n-1}(x_{n-1})dx_{n-1} \cdots$$

$$\int_0^{x-x_n-\cdots-x_2} f_1(x_1)f_0(x - x_n - \cdots - x_2 - x_1)dx_1$$

$$= L^{-1}\{Lf_0(x) \cdot Lf_1(x) \cdots Lf_n(x)\}.$$

Now take

$$f_r(x) = x^{\alpha_r - 1} \qquad (r = 0, 1, \cdots, n),\ \alpha_r > 0. \tag{2}$$

Then

$$\int_0^x x_n^{\alpha_n - 1}dx_n \int_0^{x-x_n} x_{n-1}^{\alpha_{n-1}-1}dx_{n-1} \cdots$$

$$\int_0^{x-x_n-\cdots-x_2} x_1^{\alpha_1-1}(x - x_1 - \cdots - x_n)^{\alpha_0-1}dx_1 = L^{-1}\left\{\frac{\Gamma(\alpha_0)}{p^{\alpha_0}} \cdot \frac{\Gamma(\alpha_1)}{p^{\alpha_1}} \cdots \frac{\Gamma(\alpha_n)}{p^{\alpha_n}}\right\}$$

$$= \frac{\Gamma(\alpha_0)\Gamma(\alpha_1) \cdots \Gamma(\alpha_n)}{\Gamma(\alpha_0 + \alpha_1 + \cdots + \alpha_n)} x^{\alpha_0+\alpha_1+\cdots+\alpha_n-1}.$$

When $x = 1$ the above relation reduces to

$$\int_0^1 x_n^{\alpha_n-1}dx_n \int_0^{1-x_n} x_{n-1}^{\alpha_{n-1}-1}dx_{n-1} \cdots$$

$$\cdot \int_0^{1-x_n-\cdots-x_2} x_1^{\alpha_1-1}(1 - x_1 - x_2 - \cdots - x_n)^{\alpha_0-1}dx_1 = \frac{\Gamma(\alpha_0)\Gamma(\alpha_1) \cdots \Gamma(\alpha_n)}{\Gamma(\alpha_0 + \alpha_1 + \cdots + \alpha_n)}$$

i.e.,

$$\iint_{R_n} \cdots \int x_1^{\alpha_1-1}x_2^{\alpha_2-1} \cdots x_n^{\alpha_n-1}(1 - x_1 - \cdots - x_n)^{\alpha_0-1}dx_1dx_2 \cdots dx_n$$

$$= \frac{\Gamma(\alpha_0)\Gamma(\alpha_1) \cdots \Gamma(\alpha_n)}{\Gamma(\alpha_0 + \alpha_1 + \cdots + \alpha_n)}.$$

When $n = 1$ this can be written

$$\int_0^1 x_1^{\alpha_1-1}(1 - x_1)^{\alpha_0-1}dx_1 = \frac{\Gamma(\alpha_0)\Gamma(\alpha_1)}{\Gamma(\alpha_0 + \alpha_1)}; \quad \textit{i.e.}, \quad B(\alpha_1, \alpha_0) = \frac{\Gamma(\alpha_0)\Gamma(\alpha_1)}{\Gamma(\alpha_0 + \alpha_1)}.$$

If instead of (2) we choose

$$f_r(x) = x^{\alpha_r - 1} \quad (r = 1, \cdots, n), \qquad f_0(x) = f(x) = g(1 - x)$$

then we obtain in the same way the more general relation

$$\iint_{R_n} \cdots \int x_1^{\alpha_1-1} x_2^{\alpha_2-1} \cdots x_n^{\alpha_n-1} g(x_1 + x_2 + \cdots + x_n) dx_1 \cdots dx_n$$

$$= \frac{\Gamma(\alpha_1)\Gamma(\alpha_2) \cdots \Gamma(\alpha_n)}{\Gamma(\alpha_1 + \alpha_2 + \cdots + \alpha_n)} \int_0^1 t^{\alpha_1+\alpha_2+\cdots+\alpha_n-1} g(t) dt.$$

CLASSROOM NOTES

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A NATURAL APPROACH TO THE FUNDAMENTAL THEOREM OF THE INTEGRAL CALCULUS

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A great deal of the work of elementary mathematics is concerned with operations and functions which are inverses of each other. One of the things to emphasize in teaching mathematics is how the knowledge of one of two inverse operations or inverse functions facilitates the use of the other. With this emphasis in mind, it is natural to see if summations can be effected by means of differences. That such is the case is well known to students of the calculus of finite differences but I have not seen this technique carried over to the calculus (of infinitesimal differences). Without discussing whys and wherefores, suffice it to say that I have used the following procedure to help first year students appreciate the fundamental theorem of the integral calculus.

We will assume that the student is familiar with the usual concepts and formulae that precede the fundamental theorem except that he need not to have been exposed to differentiation of the trigonometric, logarithmic, or exponential functions. In particular we will assume that he knows

(1) If $f(x) = x^n$, $\Delta f(x) = f'(x)\overline{\Delta x} + f''(x)\dfrac{\overline{\Delta x}^2}{2!} + \cdots + f^{(n)}(x)\dfrac{\overline{\Delta x}^n}{n!}$, n a positive integer.

(2) If $f(x) = x^n$, $df(x) = f'(x)\Delta x = \Delta f(x) - o(\Delta x)$, n a positive integer.

Statement (1) is useful as giving a preview of Maclaurin's theorem and also as a practical way of finding Δx^n easily.

We then show that if the closed interval (a, b) is divided into n sub-intervals by the points $a=x_0<x_1<x_2<\cdots<x_n=b$ that

$$\sum \Delta x_i = \sum (x_{i+1} - x_i) = (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1}) = x_n - x_0 = b - a. \tag{3}$$

(Here, as in all uses of $\sum$ that follow, we assume the summation is from $i=0$ to $i=n-1$ unless otherwise noted.) Statement (3) is true regardless of the manner of division, but from this point on we will assume the sub-intervals are of equal length so that $\Delta x_{i+1}=\Delta x_i$.

More generally, if $f(x)$ is a function continuous throughout the closed interval (a, b) and if the interval is divided as above,

$$\sum \Delta f(x_i) = f(x_n) - f(x_0) = f(b) - f(a), \qquad \text{for any } n. \tag{4}$$

We illustrate this last statement to the class by taking a function such as x^3 over a small interval such as (2, 6).

We then point out that the simple type of summation given in (4) is rarely encountered in mathematical applications but that the two types which are most frequently encountered, namely $\sum f(x_i)$ and the limit of $\sum f(x_i)\Delta x_i$ as n increases without limit can often be evaluated by (4). Evaluation of the latter type is the main concern of the integral calculus and we now show by considering a special case how this can be effected.

Suppose we wish to find (this can be motivated in the usual manner by "area under a curve")

$$\lim_{n\to\infty} \sum 3x_i^2\Delta x_i, \qquad x_0 = 2, \qquad x_n = 6, \qquad \Delta x_i = x_{i+1} - x_i = \frac{6-2}{n}. \tag{5}$$

(In all summations which follow, unless otherwise indicated, $x_0=2$, $x_n=6$, $\Delta x_i=x_{i+1}-x_i=(6-2)/n$.)

Now we know that $dx^3=3x^2\Delta x$ while $\Delta x^3=3x^2\overline{\Delta x}+3x\overline{\Delta x}^2+\overline{\Delta x}^3$ (this latter is not quickly recognized by the average student but is easily found by (1)).

From (4) we know that

$$\sum \Delta x_i^3 = x_n^3 - x_0^3 = 6^3 - 2^3 \tag{6}$$

for any integral n. Hence

$$\sum 3x_i^2\overline{\Delta x_i} + \sum 3x_i\overline{\Delta x_i}^2 + \sum \overline{\Delta x_i}^3 = 6^3 - 2^3 \tag{7}$$

for any integral n.

Hence

$$\lim_{n\to\infty} \left[\sum 3x_i^2\overline{\Delta x_i} + \sum 3x_i\overline{\Delta x_i}^2 + \sum \overline{\Delta x_i}^3\right] = 6^3 - 2^3. \tag{8}$$

It is easily shown, without using any summation formulae, that

$$\lim_{n\to\infty} \left[\sum 3x_i\overline{\Delta x_i}^2 + \sum \overline{\Delta x_i}^3\right] = 0 \tag{9}$$

so that

$$\lim_{n\to\infty} \sum 3x_i^2\overline{\Delta x_i} = \sum \Delta x_i^3 = 6^3 - 2^3. \tag{10}$$

In (10) we have a special case of the fundamental theorem which we may express symbolically as

$$\lim_{n\to\infty} \sum df(x_i) = \sum \Delta f(x_i) = f(x_n) - f(x_0) \tag{11}$$

or, introducing the integral notation, as

$$\lim_{n\to\infty} \sum df(x_i) = \sum \Delta f(x_i) = \int_{x_0}^{x_n} df(x) = f(x_n) - f(x_0). \tag{12}$$

As an illustration of the finite type of summation we might use the following:

$$\text{Find } 2^2 + 2.5^2 + 3^2 + 3.5^2 + \cdots + 51.5^2. \tag{13}$$

In our summation notation, this is written

$$\sum_{i=0}^{99} x_i^2, \qquad x_0 = 2, \qquad x_n = 52, \qquad x_{i+1} - x_i = \tfrac{1}{2}, \qquad n = 100. \tag{14}$$

To express this in a form for which (4) and (12) are applicable, we divide and multiply by $3\Delta x_i$ where $\Delta x_i = \frac{1}{2}$ so that

$$\sum x_i^2 = \frac{2}{3}\sum 3x_i^2\Delta x_i = \frac{2}{3}\left[\sum \Delta x_i^3 - \sum 3x_i\overline{\Delta x_i}^2 - \sum \overline{\Delta x_i}^3\right] \tag{15}$$

$$= \frac{2}{3}\left[\int_2^{52} 3x^2dx - \frac{3}{2}\overline{\Delta x}\int_2^{52} 2xdx + \frac{1}{2}\sum \overline{\Delta x_i}^3\right] \tag{16}$$

$$= 92{,}387.5.$$

In like manner we can derive the general formula for the sum of n squares

$$a^2 + (a+d)^2 + (a+2d)^2 + \cdots + b^2 = abn + \frac{nd^2}{6}(n-1)(2n-1), \tag{17}$$

or the formula for the sum of n mth powers, and even the Euler-Maclaurin summation formula.

A DERIVATION OF sin $(A-B)$ AND cos $(A-B)$

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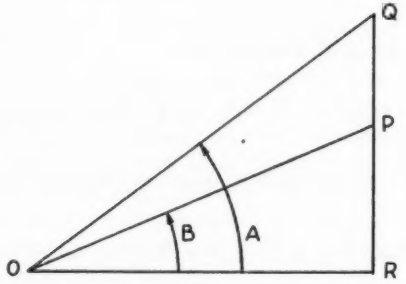
Since most texts use a rather tricky diagram to derive the formulas for the sin $(A+B)$ and cos $(A+B)$, it might be worth while to first derive the formulas for sin $(A-B)$ and cos $(A-B)$, and from these obtain the others. The derivation which follows assumes that the sine and cosine laws have been previously covered, and such is usually the case. A distinct advantage in the following method, other than the absence of any auxiliary lines, is the fact that it is natural to derive the sin $(A-B)$ by means of the sine law, and the cos $(A-B)$ by means of the cosine law. I'm sure it would be much more easily remembered by students.



By sine law in triangle OPQ

$$\frac{\sin(A-B)}{QP}=\frac{\sin\left(\frac{\pi}{2}-A\right)}{OP}=\frac{\cos A}{OP}$$

$$\sin(A-B)=\frac{QP\cos A}{OP}$$

$$=\frac{(QR-PR)\cos A}{OP}$$

$$=\frac{QR\cos A}{OP}-\frac{PR}{OP}\cos A$$

$$=\frac{QR}{OP}\cdot\frac{OR}{OQ}-\sin B\cos A$$

$$=\frac{QR}{OQ}\cdot\frac{OR}{OP}-\sin B\cos A$$

$$=\sin A\cos B-\sin B\cos A$$

By cosine law in triangle OPQ

$$\cos(A-B)$$

$$=\frac{\overline{OP}^2+\overline{OQ}^2-\overline{QP}^2}{2\overline{OP}\,\overline{OQ}}$$

$$=\frac{\overline{OP}^2+\overline{OQ}^2-(\overline{QR}-\overline{PR})^2}{2\overline{OP}\,\overline{OQ}}$$

$$=\frac{\overline{OP}^2-\overline{PR}^2+\overline{OQ}^2-\overline{QR}^2+2\overline{QR}\,\overline{PR}}{2\overline{OP}\,\overline{OQ}}$$

$$=\frac{\overline{OR}^2+\overline{OR}^2+2\overline{QR}\,\overline{PR}}{2\overline{OP}\,\overline{OQ}}$$

$$=\frac{\overline{OR}}{\overline{OQ}}\cdot\frac{\overline{OR}}{\overline{OP}}+\frac{\overline{QR}}{\overline{OQ}}\cdot\frac{\overline{PR}}{\overline{OP}}$$

$$=\cos A\cos B+\sin A\sin B.$$

EXPLAINING THE LINE OF POSITION

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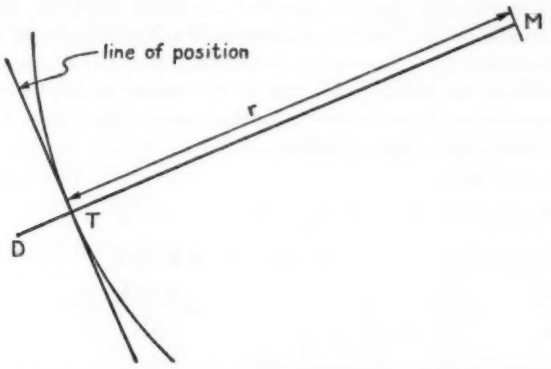
In principle, a circle of position can be obtained by drawing on a globe a circle with center at the geographical position of a star and radius equal to the star's zenith distance. This fact is easily grasped by a student at the beginning of his study of celestial navigation; but it is often not easy for him to see how it is related to the methods used in practice for drawing a line of position.

If we had a large globe and suitable drawing instruments, we could carry out the construction directly. (This has been suggested as a practical method [*Fortune*, January 1943].) If we tried to do the same thing on a large flat chart, the circle of position would usually not appear as a circle on the chart. This, however, is a minor difficulty, because we need only the part of the circle of position which passes near our approximate position. This part will, except for negligible errors, be represented by part of a circle of the same center and radius. A more serious difficulty is that to get both the center of the circle and our position on the same chart, we need either an inconveniently large chart, or one on too small a scale for accurate work. If we use, as we do in practice, a small chart on a large scale, we have to draw on the chart an arc of a circle whose radius is known and whose center is a known but inaccessible point.



To draw a circle of known radius r and known but inaccessible center M, we can take any convenient auxiliary point D on our chart, and determine the direction of M from D and the distance MD (these are the azimuth and zenith distance of the star whose geographical position is M, obtained by solving the astronomical triangle). Suppose that $MD > r$. Then a line drawn from D in the direction of M will be a radius (produced) of the circle of position. The circle will meet MD at a point T such that $DT = MD - r$. (DT is the altitude intercept.) Since in most cases the part of the circle of position which is on our chart will practically coincide with the tangent to the circle at T, we obtain the line

of position by drawing a line through T perpendicular to DM. If $MD<r$, $DT=r-MD$.

In this presentation, the "assumed position" (point D) takes its proper place as merely an auxiliary and arbitrary point. The discussion of "what we should see if we were at the assumed position," appearing in some textbooks, is an unnecessary complication.

The method just explained, that of St. Hilaire, depends on the fact that a point and a direction determine a straight line. A straight line is also determined by two points; it is this determination which is used in Sumner's original method for drawing a line of position.

RELATIVE MAXIMA AND MINIMA OF FUNCTIONS OF TWO OR MORE VARIABLES

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Of our current calculus texts that include a treatment of partial derivatives, a number fail to include any discussion of relative extremes of functions of two variables. Some discuss only necessary conditions. Others add to necessary conditions a statement without proof of sufficient conditions. And a few to be sure do present sufficient conditions with a proof usually based on Taylor's theorem. Every text and every first course, however, certainly includes both necessary and sufficient conditions for relative extremes of functions of one variable. The purpose of this paper is to present an elementary derivation, that should have wide appeal among beginning students, of the usual sufficient conditions for relative extremes of functions of two variables, and to generalize this derivation to functions of more than two variables.

We take as our basis

THEOREM 1. *If*:

1. $f(x)$ *belongs to* C^2, $a \leqq x \leqq b$,
2. $f'(c) = 0$, $a < c < b$,
3. $f''(c) > 0$,

then $f(c)$ *is a relative minimum value of* $f(x)$.

DEFINITION 1. $f(a, b)$ *is a relative minimum value of* $f(x, y)$ *in a region* R *if and only if* $f(a, b) < f(x, y)$ *for all* (x, y) *in* R *satisfying the inequality* $0 < (x-a)^2 + (y-b)^2 < \delta$ *for some positive* δ.

If $f(x, y)$ belongs to C^2 in some region R and (a, b) is an interior point of R then

$$F(s) = f(a + s \cos \alpha, b + s \sin \alpha)$$

belongs to C^2, $-d \leqq s \leqq d$.

$$F'(s) = f_1 \cos \alpha + f_2 \sin \alpha.$$

$$F''(s) = f_{11} \cos^2 \alpha + 2f_{12} \cos \alpha \sin \alpha + f_{22} \sin^2 \alpha.$$

Here the subscripts denote partial derivatives; in particular, since $f(x, y)$ belongs to C^2, $f_{12}=f_{21}$. If $f_1(a, b)=f_2(a, b)=0$, then $F'(0)=0$. $F''(0)>0$ if

$$f_{11}(a, b) \cos^2 \alpha + 2f_{12}(a, b) \cos \alpha \sin \alpha + f_{22}(a, b) \sin^2 \alpha > 0.$$

For $\sin \alpha=0$ this inequality can only be satisfied if $f_{11}(a, b)>0$; for $\sin \alpha \neq 0$ we must have in addition

$$[f_{12}(a, b) \sin \alpha]^2 - f_{11}(a, b) \cdot f_{22}(a, b) \sin^2 \alpha < 0.$$

Hence we have

THEOREM 2. *If*:

1. $f(x, y)$ *belongs to* C^2 *in some region* R,
2. $f_1(a, b) = f_2(a, b) = 0$, (a, b) *an interior point of* R,
3. $f_{11}(a, b) > 0$,
4. $f_{11}(a, b)f_{22}(a, b) - f_{12}^2(a, b) > 0$,

then $f(a, b)$ *is a relative minimum value of* $f(x, y)$ *in* R.

For appropriate functions $f(x_1, x_2, x_3, \cdots, x_n)$ we make the transformation

$$x_i = a_i + sh_i \qquad \text{where } \sum_{i=1}^{n} h_i^2 = 1.$$

Also, we shall use the notation f_i and f_{ij} for the various derivatives of f evaluated at $x_i=a_i$. Thus:

$$F(s) = f(a_1 + sh_1, a_2 + sh_2, \cdots, a_n + sh_n)$$

$$F(0) = f(a_1, a_2, a_3, \cdots, a_n)$$

$$F'(0) = \sum_{i=1}^{n} f_i h_i$$

$$F''(0) = \sum_{i,j=1}^{n} f_{ij} h_i h_j.$$

$F'(0)=0$ for all sets of values h_i if $f_i=0$, $(i=1, 2, 3, \cdots, n)$.

For $F''(0)=\sum_{i,j=1}^{n} f_{ij}h_ih_j$ to be greater than zero we must have:

1. If $h_i=0$, $(i=2, 3, 4, \cdots, n)$, $f_{11}>0$;
2. If $h_i \neq 0$, $(i=1, 2, 3, \cdots, n)$ then, since

$$\sum_{i,j=1}^{n} f_{ij}h_ih_j = f_{11}h_1^2 + 2h_1 \sum_{j=2}^{n} f_{1j}h_j + \sum_{i,j=2}^{n} f_{ij}h_ih_j,$$

we must have in addition to $f_{11}>0$,

$$f_{11}\sum_{i,j=2}^{n} f_{ij}h_ih_j - \left[\sum_{j=2}^{n} f_{1j}h_j\right]^2 = [f_{11}f_{22} - f_{12}^2]h_2^2$$

$$+ 2\left[f_{11}\sum_{j=3}^{n} f_{2j}h_j - \sum_{k=3}^{n} f_{12}f_{1k}h_k\right]h_2 + f_{11}\sum_{i,j=3}^{n} f_{ij}h_ih_j - \sum_{j,k=3}^{n} f_{1j}f_{1k}h_jh_k > 0.$$

If $h_i=0$, $(i=3, 4, \cdots, n)$, or if $n=2$, we must have then

$$f_{11}f_{22} - f_{12}^2 > 0.$$

3. If $h_i\neq 0$, $(i=1, 2, 3, 4, \cdots, n)$, and $n>2$, we must have in addition

$$[f_{11}f_{22} - f_{12}^2]\left[f_{11}\sum_{i,j=3}^{n} f_{ij}h_ih_j - \sum_{j,k=3}^{n} f_{1j}f_{1k}h_jh_k\right]$$

$$- \left[f_{11}\sum_{j=3}^{n} f_{2j}h_j - \sum_{k=3}^{n} f_{12}f_{1k}h_k\right]^2 > 0.$$

If $h_i=0$, $(i=4, 5, 6, \cdots, n)$, or if $n=3$, we must have

$$[f_{11}f_{22} - f_{12}^2][f_{11}f_{33} - f_{13}^2] - [f_{11}f_{23} - f_{12}f_{13}]^2 > 0.$$

We have thus established

THEOREM 3. *If*:

1. $f(x, y, z)$ *belongs to* C^2 *in some region* R,
2. $f_1=f_2=f_3=0$ *at* (a_1, a_2, a_3) *an interior point of* R,
3. $f_{11}>0$ *at* (a_1, a_2, a_3),
4. $\begin{vmatrix} f_{11} & f_{21} \\ f_{12} & f_{22} \end{vmatrix} > 0$ *at* (a_1, a_2, a_3),
5. $\begin{vmatrix} f_{11} & f_{21} & f_{31} \\ f_{12} & f_{22} & f_{32} \\ f_{13} & f_{23} & f_{33} \end{vmatrix} > 0$ *at* (a_1, a_2, a_3),

then $f(a_1, a_2, a_3)$ *is a relative minimum value of* $f(x, y, z)$.

We are now in position to derive the corresponding theorems for functions of four or more variables.

The above can easily be adapted to maximum values by the reversal of appropriate inequality signs.

SOME STANDARD PROBLEMS IN INTEGRATION SIMPLIFIED

D. G. DUNCAN, University of Arizona

Students are often confused by the first applications of the fundamental theorem of the integral calculus, simply because the first problems they encounter involve either integrals with which they are not sufficiently familiar, or situations in which the basic idea of integration is lost in the details of analytical geometry required to set up the problems.

In the following examples, the intrinsic geometry of the figure involved, rather than an imposed coordinate system, is used. This leads to simpler and more intuitive solutions than those usually found in the texts.

(a) The standard derivation of the formula for the area of a circle involves the integral: $\int_0^a y\,dx = \int_0^a \sqrt{a^2 - x^2}\,dx$.
However, by an obvious application of the fundamental theorem to a different element of integration we have: $A = \int_0^R 2\pi r\,dr = \pi R^2$.

(b) An integral for determining the area of a sphere is easily set up with the aid of Fig. 1; it is:

$$A = 2\int_0^{\pi/2} 2\pi R \sin\theta (R\,d\theta) = 4\pi R^2.$$

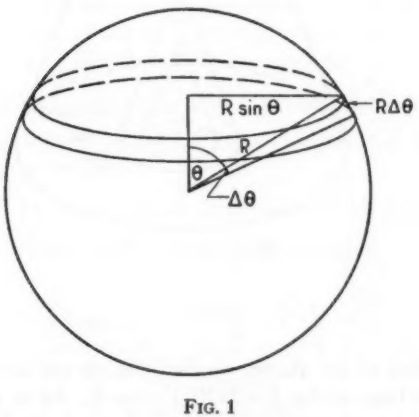

FIG. 1

(c) If we think of a sphere as composed of a number of concentric spherical shells of thickness Δr, its volume is determined immediately by the integral: $V = \int_0^R 4\pi r^2\,dr = \frac{4}{3}\pi R^3$.

(d) From the formula for the area of the lateral surface of a right circular cone, $A = \pi r s$ (s = slant height), the volume of a cone is easily computed

by taking concentric conical shells of thickness Δr $(\cos \alpha)$ as the elements of integration (Fig. 2).

$$V = \int_0^R (\pi r s)(\cos \alpha dr)$$

$$= \int_0^R \pi r^2 \cot \alpha dr$$

$$= \tfrac{1}{3}\pi R^3 \cot \alpha = \tfrac{1}{3}\pi R^2 H.$$

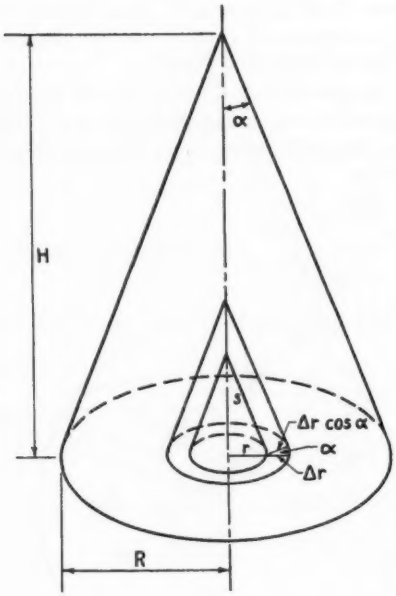

FIG. 2

(e) By the method of (b) above, we find the curved area of a spherical segment of one base to be $A = 2\pi R^2(1-\cos \theta)$. As in (c) above, this area may be used to calculate the volume of a spherical cone:

$$V = \int_0^R 2\pi r^2(1-\cos \theta)\, dr = \tfrac{2}{3}\pi R^3\,(1-\cos \theta).$$

Examples of the above type invariably help the student to visualize the process of integration and to cope with problems he may meet in the physical sciences which involve integration but do not correspond to a specific text-book situation.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Harpur College, Endicott, New York. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1121. *Proposed by W. D. Serbyn, Carnegie Institute of Technology*

Let E be any collection of n integers, not necessarily distinct. Show that there exists a non-empty subcollection $F \subseteq E$ such that the sum of the integers contained in F is divisible by n.

E 1122. *Proposed by P. B. Johnson, Occidental College and Haverford College*

Perfectly rigid playing cards are piled on the edge of a table with the pile slanting up away from the table. How far from the edge of the table can the pile be made to extend without falling to the floor?

E 1123. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Given a square $N \times N$ point lattice, show that it is possible to draw a polygonal path passing through all the N^2 lattice points and consisting of $2N-2$ segments. Can it be done with less than $2N-2$ segments?

E 1124. *Proposed by L. J. Lander and J. L. Selfridge, U. C. L. A.*

Let $(a; 0) = 1$ and $(a; n+1) = a^{(a;n)}$. Find all solutions of $(a; m) = (b; n)$ in integers a, b, m, n all greater than 1.

E 1125. *Proposed by Walter James, University of Minnesota*

Determine the coefficients B_j^p for the following sum:

$$\sum_{s=1}^{n} s^p = \frac{1}{(n-1)!} \sum_{j=1}^{p} \frac{B_j^p (n+j)!}{j+1}.$$

SOLUTIONS

Series of Reciprocals

E 1091 [1953, 711]. *Proposed by J. Lambek, McGill University, and Leo Moser, University of Alberta*

Given a sequence of integers $n_1 < n_2 < \cdots < n_i < \cdots$ such that for $j > i$ the decimal representation of n_j does not begin (on the left) with the decimal representation of n_i. Prove that

$$\sum_i 1/n_i \leqq 1 + 1/2 + 1/3 + \cdots + 1/9.$$

Solution by P. B. Johnson, Occidental College and Haverford College. Obviously one can assume that the n_i are positive. If the integers 1, 2, 3, $\cdots$, 9 are all in the set, there can be no others, and equality follows. To fix our ideas, suppose 3 is not in the set. Successive inequalities of the type $1/30+1/31+1/32+\cdots+1/39<10/30=1/3$ shows that the sum of the reciprocals of all allowable numbers beginning with 3 is less than 1/3. Similar argument shows that the sum of the reciprocals of all permissible numbers beginning with $k<10$ is less than $1/k$. Hence the desired result.

Also solved by J. W. Baldwin, J. L. Botsford, W. E. Briggs, A. R. Hyde, H. N. Laden, D. C. B. Marsh, Joseph Muskat, L. A. Ringenberg, and H. F. Trotter.

The solution and result are easily generalized for any base of representation. For base 2 we find that the sum of reciprocals of a satisfactory sequence is at most 1.

Orthic and Tangential Triangles

E 1092 [1953, 711]. *Proposed by N. A. Court, University of Oklahoma*

The homothetic center of the orthic and tangential triangles of a given triangle (T) (see the proposer's *College Geometry*, 2nd ed., p. 98, art. 191) is the pole of the orthic axis of (T) with respect to the circumcircle of (T).

Solution by the Proposer. Let the sides BC, CA, AB of $(T)=ABC$ meet the orthic axis $D'E'F'$ of (T) in the points D', E', F', respectively. The polar of the point D' with respect to the circumcircle (O) of (T) passes through the harmonic conjugate of D' with respect to the points B, C of (O), which point is the foot D of the altitude AD of (T) (*ibid*, p. 241, ex. 1). The polar of D' for (O) also passes through the pole A'' of the side BC with respect to (O). Thus the polar of D' is the line $A''D$ which joins the vertex A'' of the tangential triangle $A''B''C''$ of (T) to the vertex D of the orthic triangle DEF of (T).

Now the point D' lies on the orthic axis $D'E'F'$, hence $A''D$ passes through the pole L of $D'E'F'$ for the circle (O). Similarly the polars $B''E$, $C''F$ of the points E', F' for (O) pass through the point L. Hence the proposition.

Also solved by G. B. Charlesworth, Joseph Langr, and Victor Thébault.

Thébault pointed out that the ratio of homothety is $2R\cos A\cos B\cos C$.

Limit of a Sequence

E 1093 [1953, 711]. *Proposed by H. S. Wilf, Nuclear Development Associates, White Plains, N. Y.*

Define $S_0=1$, $S_1=3$, $S_{n+1}=2S_n^2-1$ for $n\geqq 1$. Find

$$\lim_{n\to\infty}\frac{S_n}{2^n S_0 S_1\cdots S_{n-1}}.$$

I. *Solution by D. C. B. Marsh, University of Colorado.* From

$$S_n^2 - 1 = (S_n + 1)(S_n - 1) = 2S_{n-1}^2(2S_{n-1}^2 - 2) = 2^2S_{n-1}^2(S_{n-1}^2 - 1),$$

it follows that

$$S_n^2 - 1 = 2^{2(n-1)}S_{n-1}^2S_{n-2}^2 \cdots S_1^2(S_1^2 - 1),$$

whence

$$\frac{S_n}{2^nS_0S_1 \cdots S_{n-1}} = \frac{S_n}{2S_0\sqrt{\dfrac{S_n^2 - 1}{S_1^2 - 1}}} = \frac{\sqrt{S_1^2 - 1}}{2S_0}\sqrt{\frac{S_n^2}{S_n^2 - 1}},$$

which has as limit

$$\sqrt{(S_1^2 - 1)/2S_0} = \sqrt{2}.$$

II. *Solution by J. V. Whittaker, University of California, Los Angeles.* Let r be the positive root of the equation $\cosh x = 3$. Since $\cosh 2x = 2\cosh^2 x - 1$, we have $S_n = \cosh 2^{n-1}r$. Then

$$\begin{aligned}\frac{S_n}{2^nS_1 \cdots S_{n-1}} &= \frac{\cosh 2^{n-1}r}{2^n \cosh r \cdots \cosh 2^{n-2}r} \\ &= (1/2)\sinh r \coth 2^{n-1}r \\ &= \sqrt{2}\coth 2^{n-1}r.\end{aligned}$$

As $n \to \infty$, this quantity approaches $\sqrt{2}$, since $\coth x \to 1$ as $x \to \infty$.

Also solved by R. P. Bailey, J. W. Baldwin, L. F. Boron, J. L. Botsford, Bernice Brown, G. B. Charlesworth, F. J. Duarte, M. P. Epstein, H. M. Feldman, Harry Goheen, A. S. G. Grant, A. R. Hyde, M. S. Klamkin, Viktors Linis, C. F. Pinzka, D. C. Russell, Paul Schillo, M. R. Spiegel, O. E. Stanaitis, Chih-yi Wang, L. E. Ward, Jr. and L. E. Ward, Sr. (jointly), and the proposer.

Construction of a Function

E 1094 [1953, 712]. *Proposed by Azriel Rosenfeld, Columbia University*

(1) Construct a function defined everywhere on a closed (or open) interval which takes on each of its values exactly twice on this interval.

(2) Prove that no such function can be continuous.

Solution by the Proposer. (1) It is trivially easy to construct such a function on a *half*-open interval. A one-one mapping from this to the given closed (open) interval (along the lines, for example, of the example given in Kamke, *Theory of Sets*, pp. 14–15) will give the desired function.

(2) This is a special case of the following: A continuous function which takes on no value more than twice must take on some value exactly once.

Proof: Obviously the function cannot be even locally constant. Let I be some closed interval in which it is defined, and M the maximum value which the function f takes on I. If $f=M$ nowhere else, we are done; if not, let m be the minimum which f takes on between the two points where $f=M$. Then evidently between these two points f takes on all the values $m<a\leqq M$ at least twice, and thus, by hypothesis, exactly twice. But the value m is taken on only once, since for it to be taken on twice, either within or without the interval between the two maxima, f would have to take on some of the values $m<a<M$ still more times.

Also solved by Trevor Barker, J. L. Botsford, Vern Hoggatt, H. D. Lipsich and G. M. Merriman (jointly), J. D. Miller, Norman Miller, L. L. Pennisi, L. A. Ringenberg, W. L. Shepherd, O. E. Stanaitis, G. H. M. Thomas, and L. E. Ward, Jr.

Many interesting examples of functions satisfying (1) were offered. Lipsich and Merriman supplied examples showing that for such a function monotonicity on a range of continuity is not necessary, and also that boundedness, bounded variation, and integrability are not necessary either. They also produced an example of a function defined everywhere on a closed interval which takes on each of its values exactly k times on this interval, where k is any positive integer.

Rebuses

E 1095 [1953, 712]. *Proposed by Leon Bankoff and C. W. Trigg, Los Angeles, Calif.*

Translate each of the following sketches into a mathematical term. (For the sketches, see the proposal, this MONTHLY, vol. 60, p. 712.)

Contributions by Julian Braun, Bernice Brown, P. L. Chessin, A. R. Hyde, M. S. Klamkin, D. C. B. Marsh, Norman Miller, Walter Penny, and the proposers.

1. osculating circles, osculation, triple contact, infinity, degenerate curves, elliptical (lip-tickle)

2. reflection, analysis situs, concentration, positive definite form, image, rational function, homogeneous (homo genius), everywhere dense

3. inverse sine (sign in verse), negative sign, sine law, versine, ordered field, natural barrier, closed region

4. primitive triangle, primitive solution, triad, opposition, 2:1 ratio, elementary operations, addition and subtraction, substitution, law of the mean, intercept, transformation

5. pentagon (tag on pen), boundary values, null class, pencil (pen+sell), reduction to lower terms, reduced cubic

6. moment of inertia, imbedded, curl, nappe (nap), horizontal plane, partial covering, singular

7. point set, high point contact, tangent (tan gent), points in a plane, needle problem, fixed points, multiply connected surfaces, singular points

8. asymptotes (ass totes imp), amount, develop (devil up), first law of the mean

9. matrix (May+tricks), magic square, matrix inversion (if left to right is normal order), proof by mathematical induction (like pulling rabbits out of a hat)

10. absolutely convergent, positively decreasing, convergent, bounded and convergent

11. evolution (ape+missing link+man), linkage, open interval, discontinuity, infinite descent, discontinuous chain, evolute, tractrix (evolute of a catenary), law of excluded middle

12. combination (comb+eye+nation), ordered set (I use a [U.S.A.] comb).

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4593. *Proposed by S. W. Golomb, Harvard University*

Let $0, a_1, a_2, \cdots, a_{p-1}$ be any complete residue system modulo the odd prime p. Show that

$$0, a_1, 2a_2, \cdots, (p-1)a_{p-1} \tag{1}$$

is never a complete residue system modulo p.

Show further that if any non-zero residue r is specified, the a_i's can be so chosen that every residue except r occurs in (1). What relation will the residue s occurring twice bear to r?

4594. *Proposed by Edgar Reich, Rand Corporation, Santa Monica, California*

If $f(z)$ is a complex-valued continuous function of the complex variable z, such that $f(z) = z$ whenever $|z| = 1$, show that $f(z)$ has at least one zero in $|z| < 1$.

4595. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York*

If

$$[xD]^n = \sum_{r=1}^{n} A_{rn} x^{n-r+1} D^{n-r+1},$$

where D is the differential operator, determine A_{rn}.

4596. *Proposed by G. G. Lorentz, Wayne University, and M. S. MacPhail, Carleton College, Ottawa*

Given a sequence s_n, let

$$\sigma_{n,p} = \frac{s_n + s_{n+1} + \cdots + s_{n+p-1}}{p}. \tag{1}$$

We call s_n almost absolutely convergent (a.a.c.) if the series

$$\sum_{p=1}^{\infty} |\sigma_{n,p} - \sigma_{n,p+1}|$$

converge uniformly for $n = 1, 2, \cdots$. Prove that:

1. A sequence s_n is a.a.c. if s_n is absolutely convergent, that is if $\sum |s_n - s_{n+1}| < \infty$.

2. If each a.a.c. sequence is absolutely summable by a method of summation $A = (a_{mn})$, then there is a bounded sequence which is absolutely A summable but is not a.a.c.

4597. *Proposed by Paul Erdös, University of Notre Dame*

Let $1 = a_1 < a_2 < \cdots$ be an infinite sequence of integers. Let n be any integer and write $n = a_{i_1} + a_{i_2} + \cdots + a_{i_k}$ where a_{i_1} is the greatest $a \leqq n$, a_{i_2} the greatest $a \leqq n - a_{i_1}$, *etc.* (Since $a_1 = 1$ this representation is always possible.) Put $f(n) = k$ (*i.e.*, the number of summands representing n). Prove that if the upper density of the a's is 0, then

$$\lim_{x \to \infty} \frac{1}{x} \sum_{k=1}^{x} f(k) = \infty,$$

and if the lower density of the a's is > 0, then

$$\overline{\lim_{x \to \infty}} \frac{1}{x} \sum_{k=1}^{x} f(k) < \infty.$$

(If $N(y)$ denotes the number of a's $\leqq y$, the upper density of the a's is $\overline{\lim} N(y)/y$ and the lower density is $\underline{\lim}\, N(y)/y$.)

SOLUTIONS

Optimum Strategy for an Addition Game

3651 [1933, 610]. *Proposed by Orrin Frink, Pennsylvania State University*

Two persons play a game using 24 cards numbered 1 to 6, placed face up (there being four cards of each number), by alternately turning over any card

not previously turned. The first player to make the sum of the numbers on the cards turned over exceed 30, loses. With best play, should the player who plays first always win, or always lose?

Solution by D. C. B. Marsh, Colorado University. The first player always wins if he leads with a 2 and uses the following strategy.

Initially, every card has a complement with respect to 7. A player certainly wins if his play brings the total to 30 (or to 29 if there are no 1's left). A primary strategy, then, is for the first player, A, to lead with 2 and follow each x played by B with $7-x$. It is still possible for B so to play that all 2's are exhausted before A has achieved a total of 30. Hence if B responds to the original lead of 2 with either 5 or 2, the strategy must be modified.

(1) Let the first three plays be 2-5-1. Then A should play $8-x$ to B's x at the first opportunity and $7-x$ thereafter. If B always plays 1, A should respond with 6 each time, reaching a total of 29, which wins in this case since all 1's are exhausted.

(2) If the first three plays are 2-2-2, A's responses to various possible further plays by B are best shown in a table, although the strategy in each case is quite transparent.

| B | 6 2 | | |

A Multiplication Theorem for Meromorphic Functions

4531 [1953, 193]. *Proposed by R. M. Redheffer, University of California, Los Angeles*

Let $f(z)=\sum a_n z^n$ have simple poles with positive residues at the points $d_m \neq 0$, and similarly for $g(z)=\sum b_n z^n$ at the points $e_m \neq 0$. Both functions are supposed regular elsewhere. Then the function $\sum a_n b_n z^n$ has simple poles with negative residues at the points $d_m e_n$ and no other finite singularities.

Solution by the Proposer. Define the operation $\cdot$ by $f \cdot g = \sum a_n b_n z^n$, f and g being as in the problem. Then one shows trivially

$$f \cdot g = g \cdot f, \qquad f \cdot (g + h) = f \cdot g + f \cdot h, \tag{1}$$

$$\frac{a}{z-b} \cdot f(z) = -\frac{a}{b} f\left(\frac{z}{b}\right) \qquad (a, b \text{ constant}) \tag{2}$$

$$\frac{p}{z-d} \cdot \frac{q}{z-e} = -\frac{pq}{z-de} \qquad (d, e, p, q \text{ constant}). \tag{3}$$

It suffices to get the result for an arbitrary (large) circle $|z| < R$. Let the residues of the poles for f and g be respectively f_i and g_i. We may write (making use of the hypothesis)

$$f(z) = \sum_{i=1}^{p} \frac{f_i}{z - d_i} + F(z) \tag{4}$$

where $d_1, d_2, \cdots, d_p$ are the (finite number of) poles of $f(z)$ inside the circle $|z| = R/d$ and $F(z)$ is regular for the same disk. Here d stands for the smallest of the numbers $(1, |d_i|, |e_i|)$. Writing down a relation like (4) for g and computing $f \cdot g$ we get, by (1),

$$f \cdot g = \sum \sum \frac{f_i}{z - d_i} \cdot \frac{g_i}{z - e_i} + \sum \frac{f_i}{z - d_i} \cdot G + \sum \frac{g_i}{z - e_i} \cdot F + F \cdot G.$$

By (3) the first sum has simple poles with negative residues at all points $d_m e_n$ which lie in $|z| < R$, and by (2) the next two sums are regular in this circle. Regularity of the last follows by computing the radius of convergence, which is $\geqq R/d \geqq R$. (It has been brought to the author's attention that this method is well known. See Borel, Sur les singularities des series de Taylor, *Bull. de la Soc. Mathem. de France*, t. XXVI, and Pincherle, A proposito di un recente teorema del sig. Hadamard, *Rendiconto de l'Ac. des Sc. de Bologne*. See also Hadamard's multiplication theorem, Titchmarsh, *Theory of Functions*, pp. 157–159. The sole novelty in the present treatment lies in the specialization to simple poles.)

Also solved by R. M. Breusch and O. E. Stanaitis.

Primitive Root (mod p)

4532 [1953, 193]. *Proposed by Leonard Carlitz, Duke University*

Let p be a prime >3 and let g denote a primitive root (mod p). Prove

1. $$1+\sum_{r=1}^{(p-3)/2}\frac{(1+g)\cdots(1+g^r)}{(1-g)\cdots(1-g^r)}=\prod_{r=1}^{(p-3)/2}(1+g^r)\pmod p.$$

2. If $p\equiv 3 \pmod 4$, then

$$1+\sum_{r=1}^{(p-3)/2} g^{r(r+1)/2}\frac{(1+g)\cdots(1+g^r)}{(1-g)\cdots(1-g^r)}=\prod_{r=1}^{(p-3)/4}(1+g^{2r})\pmod p;$$

if $p\equiv 1 \pmod 4$, the sum vanishes.

Solution by the Proposer. 1. We shall prove the following slightly more general result. Let $GF(p^n)$ denote a finite field of order p^n, $p>2$, and let $2m$ divide (p^n-1). Let α be a number of $GF(p^n)$ that belongs to the exponent $2m$, and let x be an indeterminate. Then

$$\prod_{r=1}^{m-1}(1+\alpha^r x)=1+\sum_{r=1}^{m-1}\frac{(1+\alpha)\cdots(1+\alpha^r)}{(1-\alpha)\cdots(1-\alpha^r)}x^r. \tag{1}$$

Proof. We recall the identity

$$\prod_{r=1}^{m-1}(1+q^r x)=1+\sum_{r=1}^{m-1}\frac{(1-q^{m-1})\cdots(1-q^{m-r})}{(1-q)\cdots(1-q^r)}q^{r(r+1)/2}x^r. \tag{2}$$

In (2) place $q=\alpha$; then, since $\alpha^m\equiv -1$, we have

$$(1-\alpha^{m-1})\cdots(1-\alpha^{m-r})=(1+\alpha)\cdots(1+\alpha^r)\alpha^{-r(r+1)/2},$$

and (1) follows.

2. Let $2m$ divide (p^n-1) and let α belong to the exponent $2m$. We shall prove

$$1+\sum_{r=1}^{m-1}\alpha^{r(r+1)/2}\frac{(1+\alpha)\cdots(1+\alpha^r)}{(1-\alpha)\cdots(1-\alpha^r)}=\begin{cases}\displaystyle\prod_{r=1}^{(m-1)/2}(1+\alpha^{2r}) & (m \text{ odd})\\ 0 & (m \text{ even}).\end{cases} \tag{3}$$

Proof. We shall use the formula of Gauss (see, for example, G. B. Mathews, *Theory of Numbers*, p. 210.)

$$1+\sum_{r=1}^{m-1}(-1)^r\frac{(1-x^{m-1})\cdots(1-x^{m-r})}{(1-x)\cdots(1-x^r)}=\begin{cases}\displaystyle\prod_{r=1}^{(m-1)/2}(1-x^{2r-1}) & (m \text{ odd})\\ 0 & (m \text{ even}).\end{cases} \tag{4}$$

In (4) place $x=\alpha^{-1}$; then, since $\alpha^m \equiv -1$, we have

$$\frac{(1-\alpha^{-m+1})\cdots(1-\alpha^{-m+r})}{(1-\alpha^{-1})\cdots(1-\alpha^{-r})} = (-1)^r \alpha^{r(r+1)/2} \frac{(1+\alpha)\cdots(1+\alpha^r)}{(1-\alpha)\cdots(1-\alpha^r)},$$

so that the left member of (4) reduces to the left member of (3). Similarly the right member of (4) yields for m odd

$$\prod_{r=1}^{(m-1)/2}(1-\alpha^{-2r+1}) = \prod_{r=1}^{(m-1)/2}(1+\alpha^{m-2r+1}) = \prod_{r=1}^{(m-1)/2}(1+\alpha^{2r}).$$

This completes the proof of (3).

The Fermat Equation

4533 [1953, 267]. *Proposed by R. Kissling, Student, University of California, Berkeley*

Given $a^n+b^n=c^n$ with a, b, c, n integers and $a>b>1$, $n\geqq 2$; $\sigma_n(k)$ being the sum of the nth powers of all divisors of k, prove

$$(1) \qquad \left|1-\frac{\sigma_n(c)}{\sigma_n(a)+\sigma_n(b)}\right| < \frac{2n-1}{n(n-1)} < \frac{2}{n-1},$$

$$(2) \qquad \lim_{k\to\infty} \frac{\sigma_k(c)}{\left\{[\sigma_k(a)]^{n/k}+[\sigma_k(b)]^{n/k}\right\}^{k/n}} = 1.$$

Solution by J. V. Whittaker, University of California at Los Angeles. From the well-known inequalities

$$a^n < \sigma_n(a) < a^n\zeta(n), \qquad \frac{2^n-1}{2^n+1} < \zeta(n) < \frac{2^n}{2^n-2},$$

it follows that

$$\frac{1}{\zeta(n)} < \frac{\sigma_n(c)}{\sigma_n(a)+\sigma_n(b)} < \zeta(n)$$

and, hence, that

$$\left|1-\frac{\sigma_n(c)}{\sigma_n(a)+\sigma_n(b)}\right| < \zeta(n)-1 < \frac{1}{2^{n-1}-1} < \frac{2n-1}{n(n-1)}.$$

Moreover,

$$\frac{1}{\zeta(k)} < \frac{\sigma_k(c)}{\left\{[\sigma_k(a)]^{n/k}+[\sigma_k(b)]^{n/k}\right\}^{k/n}} < \zeta(k).$$

Since, as $k\to\infty$, $\zeta(k)\to 1$, the middle member of the inequality does the same.

Also solved by R. R. Phelps and the Proposer.

A Permutation Matrix

4534 [1953, 268]. *Proposed by Frank Harary, University of Michigan*

If n is an odd prime and M is a symmetric n by n matrix each of whose rows is a permutation of $1, \cdots, n$, then the main diagonal is also such a permutation.

Solution by T. S. Motzkin, National Bureau of Standards, University of California, Los Angeles. Every $s=1, \cdots, n$ appears n times; of these, because of symmetry, an even number are outside the diagonal. Hence for odd n, s must appear in the diagonal and the conclusion follows.

Similarly, if a_{ijk} is an n by n by n array with permutations of $1, \cdots, n$ in all rows, and if $a_{ijk}=a_{i'j'k'}$ for any permutation $i'j'k'$ of ijk, then for n not divisible by 3 the main diagonal is also a permutation of $1, \cdots, n$.

More generally, if $a_{i_1 \cdots i_r}$ is an n^r array of permutations and "symmetric" in the same sense, and if r is a power p^t of a prime p and n not divisible by p, the main diagonal is again a permutation, by virtue of the well known fact that every multinomial coefficient $r!/\prod \alpha_\rho!$, where $r=\sum \alpha_\rho$, except $r!/r!$, is divisible by p.

Also solved by T. A. Brown, W. E. Deskins, J. E. Freund, Harry Furstenberg, Katherine Gould, S. L. Jamison, J. B. Kelly, T. C. Littlejohn and R. J. Driscoll (jointly), D. C. B. Marsh, J. H. McKay, D. A. Norton, and E. M. Wright.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio.

An Introduction to the History of Mathematics. By Howard Eves. New York, Rinehart and Company, Inc., 1953. xv+422 pages. $6.00.

There has long been a need for a book on the history of mathematics which was suitable for use as a text with undergraduates at the junior-senior level. There were a few books, but they were in turn, too advanced, too elementary, too long, too concise. Eves' book was written expressly for this purpose and this group. It succeeds remarkably well. The reviewer passes judgment not only on the basis of having read the book, but also on that of having used it as a text in a three credit hour, one semester course.

A unique and usable feature of the book is the sets of "Problem Studies" at the end of each chapter. The author contends that even in a history of mathematics class students should *do* some mathematics, not merely talk about, or, still worse, listen to others talk about it. His "Problem Studies" are not routine

problems; many outline small investigations or developments. Suggestions and answers for them are included in a twenty-five page appendix. They are designed to help the student to learn more of both mathematics and its history while having fun and accumulating enrichment material, which will, in part, be useful in secondary school teaching.

The reviewer made little use of these Problem Studies this past semester mainly because he had his own program of problems and references, but he thinks the plan excellent and well worked out. In some places the problems might be more closely related to the historical material, and perhaps more detailed and specific bibliographical material in both problems and the text would assist students. However, many of the problems do also contain significant historical material or add to the text the mathematical details needed for students to really appreciate what was done.

The book sets a high level of accuracy, both in respect to typographical errors and historical facts. Historically, the few sins are mostly of omission; for example, the sector compasses discussed on page 266 probably antedate Galileo who merely improved on the invention of Guido Ubaldo del Monte according to Gino Loria and others; the rolling wheel paradox attributed to Galileo in Problem Study 9.7 is to be found in Sir T. L. Heath's *The Mathematics in Aristotle* published posthumously in 1949. There are places, of course, where there could be honest differences of opinion or interpretation, such as the statement (page 331) that "···the so-called fundamental theorem of integral calculus is explicitly stated and proved in Barrow's *Lectiones.*"

This last is, in the reviewer's opinion, a "hindsight" error. That is, it attributes to Barrow an insight, perception, or generalization which we can now read into his work, especially if we rewrite it in modern notation, which, however, it's doubtful that Barrow really had. Barrow's chief translator and interpreter, J. M. Child, agrees with Eves, however.

In such a task as writing this book the author must make many choices as to time intervals and topics to be covered, details and persons to be included, *etc.* One can not dispute tastes, but the reviewer feels that the selection here has been well made even though he, personally, would prefer such changes as: more emphasis on a topical approach (for example, a thorough treatment of the entire story of trigonometry or of irrationals in one place at the sacrifice of interrupting the chronological progress of the book); more discussion of some more recent topics (*e.g.*, transfinite numbers, modern algebra, topology—the book essentially ends with the development of the calculus although the last ten pages are titled "Transition to the Twentieth Century"); and some treatment of mathematics in America.

The format of the book is excellent. The type is clear and well spaced, the diagrams well chosen and legible. Line drawings only have been used. There are no pictures of persons, instruments, or original works, there are no exact, and only a few approximate, excerpts from original sources. Space restrictions may not allow excursions into original sources, but by their omission there is

lost to the reader the pleasure and insight which one can derive from occasionally thinking his way into and through the processes of an original genius as he takes an important step.

In spite of these minor disagreements and suggestions, I would like to reiterate that this book accomplishes its purpose in an excellent fashion. It is not a mere recital of names and dates, but does well in an attempt to stress the growth of ideas and interrelationships between them for readers who are not too advanced or mature mathematically.

P. S. JONES
University of Michigan

Analytic Geometry and Calculus. By L. L. Smail. Appleton-Century-Crofts, Inc., New York, 1953. 84+644 pages. $5.50.

Calculus and Analytic Geometry. By G. B. Thomas, Jr. Addison-Wesley Publishing Company, Inc., Cambridge, 1953. 11+731 pages. $7.50.

Smail's text is designed for liberal arts students and for technical students. Thomas's intention in this regard is not stated explicitly, but his text appears to be designed for the latter. It may be suitable, with certain omissions, also for the former group.

Each introduces integral calculus early, before such topics as conic sections. Each contains the standard topics expected in such a book. Thomas's text, which is the larger, includes also line integrals, differentiation of vectors, scalar and vector product, and a chapter on determinants and linear equations. It contains also introductions to a number of advanced topics, notably uniform continuity, Fourier series, and functions of a complex variable. Both books conclude with a chapter on differential equations. Each has generous lists of problems. In Smail, most of the problems are arranged in pairs, and answers are supplied for the odd-numbered ones. In Thomas, all answers, including those requiring sketches, are given in an unusual forty-nine page answer section. Smail includes a table of integrals and numerical tables. Thomas does not, evidently expecting that students will purchase a set of tables. Each book is attractively printed and almost entirely free of typographical errors. Smail makes good use of bold face. Each has good three dimensional figures.

An important question in any calculus text is how the foundations are treated. In Smail's book, although there is a definition of limit given, the approach may be fairly described as intuitive. In Thomas's book, the initial approach is intuitive, and some calculus is done. Then the author proceeds to a careful development of limits, another chapter of calculus, and then to a thorough discussion of continuity. These notions are explained at length, with examples, and there are problems. The author means business. Both authors, however, made the same slip in this area. Each used without proof the continuity of linear functions. The slip is more serious in Thomas, because he was engaged at the time in proving the continuity of certain other functions. Thomas has taken the unusual step of including a discussion of uniform continuity, and an inter-

esting pedagogical device introduces the possibility that elementary calculus students will understand it. However, this work is to some extent wasted when, later, proofs of two statements about definite integrals are complete except for the unstated invocation of uniform continuity, even though the appropriate theorem (continuity in a closed interval implying uniform continuity) has been stated in the earlier section.

Smail's book is evidently the work of a skillful teacher. A number of topics which are often obscure are treated very lucidly here. Some subjects are treated with more care and precision than is customary. Careful, well-motivated definitions of work, liquid pressure, and the like are instances of this; derivations of equations of loci are others. There are some unusually good sections on infinite series. Since this is a calculus text, the reviewer would like to have seen the integral test for convergence included, but this is a matter of taste. An occasional lack of precise language was noted. The notion of something being true "generally" or "in general" changes in different contexts. At other points students may be led to believe that there is such a thing as an arbitrarily small (or large) number. A number of errors, of two kinds, were found. First there are those which occur in rather difficult topics which very few students will attempt to read thoroughly; an example is the existence proof for the derivative of the inverse function. Errors of the second kind are more serious, since they are at such a level that students may discover or be confused by them; an example of this is the second sentence of Article 236, in which the fact that the quotient of the hyperbola ordinate and a line ordinate is nearly unity is said to imply the approximate equality of the ordinates. The preceding error is not used, a correct discussion of the asymptote being supplied on the next page.

The reviewer believes that while this book has some shortcomings with respect to self-study, its many excellent qualities may make it suitable as a classroom text.

Thomas's book is a real contribution to a field in which textbooks with no essential differences are published every year. The subject matter is standard, with the additions noted above. The treatment is careful, precise, illuminating, and lively. A number of different topics are explained or illustrated in refreshingly original ways. In applications of calculus to such problems as finding the area of a surface of revolution, the question of which approximating sums are close enough to lead to the correct integral is often slighted or ignored. Here it is treated at length in a readable way which can hardly fail to add to the student's confidence when he applies calculus in new situations. The book is sprinkled with interesting and informative remarks. The problem of cutting a length of wire into two pieces, to be bent to form a circle and a square of maximum total area, is one which will give an educational jolt to cocksure students who attack such problems with a minimum of thought.

The treatment of formal integration leans heavily on hyperbolic functions and their inverses. The only form of the answer offered for $\int \sec\theta\, d\theta$ is $\sinh^{-1}(\tan\theta)+C$. In order for the student to be satisfied with this, he will need

the thorough workout with hyperbolic functions given in the preceding chapter. Early in the book, a number of problems give rise to integrals which the student is not prepared to evaluate; possibly this was intentional.

There is a tendency to use terms and symbols which are defined only subsequently, if at all; none of these omissions is serious. Three errors were noted. The first evaluation of the slope at the origin (top of page 220) involves division by zero. The three infinite expansions following (9) on page 584 are in general not valid under the stated hypotheses. The proof of l'Hospital's Theorem has a gap too difficult for a student to fill. At certain other points it was felt that a few more words of proof or discussion were necessary. The reviewer regretted that the precision which characterized the author's presentation of the foundations of elementary calculus was not retained in the discussion of vector and complex calculus.

In general, the standards of both rigor and readability were found to be very high. The reviewer's feeling is that the publication of this book represents a splendid effort to teach calculus more carefully, while simultaneously retaining, and even enhancing, its intuitive appeal.

J. H. BLAU
Antioch College

Ordinary Differential Equations. By R. E. Langer. New York, John Wiley and Sons, Inc., 1954. xii+249 pp. $4.50.

This book gives a direct and concise presentation of the topics usually treated in a beginning course on differential equations. The material on first and second order equations is presented completely, while procedures for equations of higher order are only briefly considered. A summary of special functions, such as the gamma function, Bessel functions, *etc.*, is given in the last chapter. No material on partial differential equations is included. For the most part, the applications of differential equations are presented and discussed in separate chapters, thereby allowing their exclusion for a shorter course of study.

The author has endeavored to unify the material on differential equations in as simple and clear a manner as possible. In order to do this, he has emphasized a few procedures which are not ordinarily stressed in a beginning text. This variation in some of the basic methods will be to most teachers very refreshing. Although the primary effort has been directed toward solving differential equations, the author has carefully defined all terms, and has given rigorous proofs for the existence and uniqueness of solutions to certain types of equations. His treatment of solutions by power series is especially well done.

The format of the book is decidedly excellent, and very few typographical errors were noted. The book contains some 800 problems and the answers are given for the odd-numbered problems. The book is easily adaptable to either a short or long course of study.

T. S. PETERSON
Portland State Extension Center

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

M.I.T. SPECIAL SUMMER PROGRAMS

A two-week special summer program on Mathematical Problems of Communication Theory will be presented at the Massachusetts Institute of Technology from Monday, July 12, through Friday, July 16, 1954.

The course will be under the supervision of the Department of Mathematics. Topics will include time series, Brownian motion, spectra, filtering and prediction theory, computing machines, statistical nature of communication, message and entropy, and coding. Dr. Norbert Wiener, Professor of Mathematics, and Dr. R. M. Fano and Dr. Y. W. Lee, Associate Professors in the Department of Electrical Engineering, will be the principal lecturers.

Also there will be a program in Advanced Coding Techniques for Digital Computers from August 2 to August 6 under the direction of Professor Charles W. Adams. Following this session there will be a program in the Business Applications of Digital Computers from August 16 to August 27 under the direction of Professor Adams.

SUMMER CONFERENCES FOR TEACHERS

The Second Annual Mathematics Workshop will be held by the University of Arkansas during the period June 28 to July 2, 1954. Persons interested in this institute should write to the Director, Professor Davis P. Richardson, Department of Mathematics, University of Arkansas, Fayetteville, Arkansas.

A California Conference for Teachers of Mathematics offered by University of California Extension on the Los Angeles campus of the University is set for July 6 to 16. The fourth annual event of its kind, the conference is sponsored by the University's Department of Mathematics, School of Education, and Mathematics Extension, in cooperation with the California Mathematics Council and the National Council of Teachers of Mathematics. A complete program of the conference will be mailed to those interested on request to the Department of Conferences, University of California Extension, Los Angeles 24.

SUMMER SESSIONS

The following institutions announced advanced courses in mathematics for the summer of 1954:

Columbia University. July 6 to August 13: Professor Harish-Chandra, introduction to higher algebra, algebraic topology; Professor Taylor, differential equations, higher algebra; Professor Murray, probability, theory of functions

of a complex variable; Professor Kolchin, fundamental concepts of mathematics, introduction to differential algebra; Professor Levi, topics in geometry.

The University of Chicago. June 21 to August 28: Professor Schilling, algebra IV (theory of groups, commutative rings); Professor Kaplansky, point set topology, topology I; Professor Stone, introduction to partial differential equations, Boolean algebras; Professor Halmos, topological algebra; Professor Spanier, differentiable manifolds; Dr. Auslander, cohomology theory of groups.

PERSONAL ITEMS

Associate Professor Marguerite Lehr of Bryn Mawr College and Professor I. J. Schoenberg of the University of Pennsylvania were the representatives of the Association at the Annual Meeting of the American Academy of Political and Social Science which was held in Philadelphia, Pennsylvania, on April 2–3, 1954.

The University of California announces the following: Professors Jerzy Neyman and Alfred Tarski have been invited to deliver addresses at the International Congress in Amsterdam in September, 1954; Professor C. B. Morrey, Jr. will be on sabbatical leave for the academic year 1954–55 and plans to spend this year at the Institute for Advanced Study; Professor A. L. Foster will be on sabbatical leave for the first semester of the year 1954–55; Professor E. W. Barankin and Professor M. M. Loève will also be on sabbatical leave during the year 1954–55; Professor Loève plans to spend this year in Europe.

Associate Professor A. C. Eringen of the Illinois Institute of Technology has been appointed to an associate professorship at Purdue University.

Mr. R. R. Hare, Jr. of the Air Force Missile Test Center, Patrick Air Force Base, Florida, has accepted a position as Operations Analyst with the Operations Research Office, The Johns Hopkins University, Chevy Chase, Maryland.

Miss Jane C. Ingersoll, previously a research assistant at the Los Alamos Scientific Laboratory, New Mexico, has accepted a position as Operations Analyst with the Operations Research Office, Chevy Chase, Maryland.

Mrs. Louise C. Lim, who has a Ford Foundation Fellowship for the year 1953–54, is at Radcliffe College during the second semester of this academic year.

Mr. William McKay has been appointed to an instructorship at Drexel Institute of Technology.

Professor W. R. Mann of the University of North Carolina, has been promoted to an associate professorship.

Dr. N. M. Martin, formerly of the University of California, Los Angeles, has a position as Research Associate at the Willow Run Research Center, University of Michigan.

Dr. O. B. Moan of the I.B.M. Corporation has accepted a position as quality control engineer with the Hughes Aircraft Company, Culver City, California.

Dr. J. E. Morton has been appointed Consultant on Industrial Research to

the National Science Foundation; Dr. Morton is on leave of absence from his position as Professor of Statistics at Cornell University.

Dr. R. S. Novosad of Tulane University has been appointed to an assistant professorship at Pennsylvania State College.

Dr. Rufus Oldenburger has been appointed Director of Research at the Woodward Governor Company, Rockford, Illinois.

Mr. P. B. Richards, previously an analytical engineer at the Babcock and Wilcox Research Center, Alliance, Ohio, is now a research associate at Case Institute of Technology.

Mr. Jerome Sherman, formerly an engineer at the Babcock and Wilcox Research Center, Alliance, Ohio, has accepted a position as Senior Engineer with Westinghouse Electric Corporation, Atomic Power Division, Pittsburgh, Pennsylvania.

Mr. D. E. Thoro, recently a graduate assistant at the University of Florida, is employed as a mathematician by the R.C.A. Service Company, Cocoa, Florida.

Reverend H. J. Vandort, previously curate of Grace Episcopal Church, Grand Rapids, Michigan, is now Assistant to the Bishop, Episcopal Diocese of Erie, Erie, Pennsylvania.

Dr. R. E. Zink has been appointed to an instructorship at Purdue University.

Professor Emeritus R. A. Johnson of Brooklyn College died on February 9, 1954. He was a charter member of the Association and formerly an associate editor of this MONTHLY.

Associate Professor R. F. Smith, who had retired from his position at City College of the City of New York, died on January 31, 1954.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

A VISITING LECTURESHIP PROGRAM

With the financial support of the National Science Foundation a visiting lectureship program will be administered by the Mathematical Association of America during the academic year 1954–1955. The object of the program is, by increasing the introduction of modern ideas into under-graduate mathematics,

to assist teachers in colleges and universities to improve the quality and quantity of undergraduate students in mathematics—not only those students who will eventually become research mathematicians but those who will teach it in the secondary schools, apply this subject to other fields, or merely become part of the general informed public.

With this end in view, the lecturers will be prepared not only to give formal lectures but to confer with students and faculty singly and in groups. They will be glad to advise students on future opportunities in study and employment and to discuss with members of the staff teaching problems and curriculum and throw what light they can on practices in comparable institutions. In short, the lecturers will cooperate with the departments in all ways possible toward the furtherance of the aims of the program.

Those wishing further information about the program may write to one of the following members of the committee: Professor G. B. Huff, University of Georgia, Athens, Georgia; Professor Donald E. Richmond, Williams College, Williamstown, Massachusetts; or Professor Burton W. Jones, University of Colorado, Boulder, Colorado.

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 68 persons have been elected to membership by the Board of Governors on applications duly certified.

ISRAEL ABRAMS, M.A. (Pennsylvania) Teacher, Overbrook High School, Philadelphia, Pa.

S. O. ALBERT, Student, American University.

J. E. ASHE, B.S. (Hartwick) Research Engineer, Scintilla, Sidney, N. Y.

R. W. BAGLEY, M.S. (Tulane) Instr., University of Florida.

R. A. BARNETT, M.A. (Southern California) Teaching Assistant, University of Southern California.

S. R. BEYMA, M.S. (M.I.T.) Instr., Hampton Institute.

D. W. BLAKESLEE, M. A. (California) Instr., San Francisco State College.

MRS. LOIS W. BOLAND, B.S. (Stetson) Mathematician, Patrick Air Force Base, Fla.

C. A. BROWN, M.S. (South Carolina) Asst. Professor, The Citadel.

C. J. COHEN, Ph.D. (Johns Hopkins) Mathematician, Naval Proving Ground, Dahlgren, Va.

HELEN F. CULLEN, Ph.D. (Michigan) Asst. Professor, University of Massachusetts.

R. E. DOUTT, B.S. (Grove City) Asso. Professor, South Dakota School of Mines and Technology.

G. B. GLOVER, B.S. (U.S. Military Academy) Instr., Rutgers College of South Jersey.

BASIL GORDON, M.A. (Johns Hopkins) Instr., Johns Hopkins University.

MRS. ELEANOR P. GUYER, B.A. (Hanover) Head of Department, Southport High School, Indianapolis, Ind.

A. J. HALL, Ed.D. (Stanford) Asso. Professor, San Francisco State College.

J. J. HARRIS, B.A. (Augustana) Manhattan, Kan.

PAUL HEINS, Ph.G. (Mass. Coll. of Pharmacy) Pharmacist, Thrifty Drug Co., Los Angeles, Calif.

YVONNE G. HENRION, Student, University of British Columbia.

K. L. HILLAM, B.S. (Utah) Grad. Assistant, University of Utah.

S. S. HOLLAND, JR., M.S. (Chicago) Mathematician, Army Chemical Center, Md.

J. P. HOYT, Ph.D. (George Washington) Asso. Professor, United States Military Academy.

A. ROBERTA KIEWIT, M.S. (Omaha) Chair-

man of Department, Benson High School, Omaha, Neb.

H. S. KRAMER, B.S. (U. of Washington) Analytical Statistician, United States Air Force Headquarters, Washington, D. C.

HAROLD W. KUHN, Ph.D. (Princeton) Asst. Professor, Bryn Mawr College.

H. N. LADEN, Ph.D. (Pennsylvania) Research Officer, Chesapeake & Ohio Railway Co., Cleveland, Ohio.

R. A. LAIRD, B.S.C.E. (Georgia) Lt. Col., United States Army (Retired), New Orleans, La.

LORRAINE D. LAVALLEE, B.A. (Mount Holyoke) Instr., University of Massachusetts.

R. J. LAWTHER, B.A. (Penn. State) Asso. Engineer, Douglas Aircraft Co., Los Angeles, Calif.

J. A. LECHNER, Student, Carnegie Institute of Technology.

A. B. LEHMAN, B.S. (Ohio U.) Tulane University.

C. H. LEWIS, M.S. (C.I.T.) Chairman, Division of Science and Mathematics, Orange Coast College.

H. M. LINNETTE, M.S. (Michigan) Asso. Professor, Virginia State College.

C. P. LUEHR, B.S. (Oregon S.C.) Grad. Student, Oregon State College.

MRS. DOROTHY C. MARTIN, M.A. (Peabody) Instr., Wood Junior College.

N. M. MARTIN, Ph.D. (U.C.L.A.) Research Associate, Willow Run Research Center, University of Michigan.

R. M. MASON, M.A. (Toledo) Mathematician, Naval Research Laboratory, Washington, D. C.

T. B. MATTSON, Student, Carnegie Institute of Technology.

JOHN MCCARTHY, Ph.D. (Princeton) Acting Asst. Professor, Stanford University.

J. D. MCKNIGHT, JR., Ph.D. (Purdue) Senior Aerophysics Engineer, Consolidated Vultee Aircraft Corp., Fort Worth, Tex.

L. I. MISHOE, Ph.D. (N.Y.U.) Asso. Professor of Physics, Morgan State College.

O. B. MOAN, Ph.D. (Purdue) Quality Control Staff Engineer, Hughes Aircraft Co., Culver City, Calif.

MICHAEL MONTALBANO, B.A. (George Washington) Chief, Programming Section, Computer Control Co., Point Mugu, Calif.

R. A. MORELAND, JR., B.S. (Texas Tech.) Teaching Fellow, Texas Technical College.

D. B. MUMFORD, Student, Harvard University.

S. A. NILES, B.S. (St. Francis) Instr., St. Francis College.

R. H. OWENS, Ph.D. (C.I.T.) Physical Science Coordinator, Office of Naval Research, Pasadena, Calif.

J. A. PAINTER, B.S. (Pittsburgh) Grad. Assistant, University of Pittsburgh.

E. A. PETERS, B.S. (Ohio State) 2nd Lt., United States Army.

W. W. PROCTOR, M.A. (Pennsylvania) Asst. Professor, Morgan State College.

MARGUERITE B. ROSTON, M.A. (Radcliffe) Asst. Professor, Hood College.

ELLEN A. SANDERS, M.A. (Louisville) Teacher, Taylor County High School, Campbellsville, Ky.

O. P. SANDERS, M.S. (Oklahoma A. & M.) Asst. Professor, Southeastern State College.

W. M. SANDERS, M.S. (Arkansas) Instr., Mississippi Southern College.

D. E. SANDERSON, Ph.D. (Wisconsin) Instr., Iowa State College.

SISTER M. BIBIANA, B.A. (Michigan) Instr., College of St. Teresa.

GERALDINE D. SMITH, B.A. (Bates) Editor of Mathematics Texts, Ginn and Co., Boston, Mass.

L. C. SMITH, Student, Hofstra College.

W. A. SOPER, JR., M.A. (Boston U.) Asso. Engineer, Westinghouse Electric Corp., Baltimore, Md.

G. L. SPENCER, II, Ph.D. (Michigan) Asst. Professor, University of Maryland.

GEORGE STEPHENSON, B.A. (Princeton) Research Associate, George Washington University.

J. G. SUTTON, M.A. (Harvard) Science Editor, The Macmillan Co., New York, N. Y.

REV. B. A. TONNAR, S.J., M.A. (Catholic) Asst. Prof., Loyola University, New Orleans, La.

G. D. WALDMAN, Student, Trinity College.

W. A. WHITTLESEY, II, Student, University of Oklahoma.

IZAAK WIRSZUP, M.A. (Wilno) Asst. Professor, University of Chicago.

E. J. ZINDEL, Student, Seton Hall University.

R. E. ZINK, Ph.D. (Minnesota) Instr., Purdue University.

THE JANUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The sixteenth annual meeting of the Northern California Section of the Mathematical Association of America was held at the University of San Francisco, on January 16, 1954. Professor Roy Dubisch, Chairman of the Section, presided at both the morning and afternoon sessions.

There were eighty-five persons present, including the following fifty-four members of the Association:

H. L. Alder, H. M. Bacon, G. A. Baker, T. J. Bass, Jr., E. M. Beesley, Alice K. Bell, M. T. Bird, R. L. Blair, W. E. Bleick, A. C. Burdette, L. P. Burton, J. R. Byrne, R. C. Campbell, Randolph Church, Roy Dubisch, Hazel E. Eggett, F. D. Faulkner, Ruth A. Fish, Harley Flanders, S. A. Francis, C. M. Fulton, L. C. Graue, W. H. Gregory, C. A. Hayes, Jr., J. G. Herriot, Marjorie L. Hoffman, V. E. Hoggatt, Jr., Vern James, Free Jamison, Walter Jennings, R. M. Lakness, Milton Lees, B. J. Lockhart, A. R. Lovaglia, Sophia L. McDonald, R. B. Merkel, A. B. Mewborn, H. S. Moredock, Jr., F. R. Morris, W. H. Myers, C. D. Olds, C. L. Perry, Jr., J. P. Pierce, F. M. Pulliam, E. P. Rahn, C. H. Rawlins, Jr., R. M. Robinson, E. B. Roessler, Mary V. Sunseri, Irving Sussman, Gabor Szegö, C. C. Torrance, H. G. Tucker, K. J. Waider.

At the business meeting the following officers were elected for the coming year: Chairman, Professor J. G. Herriot, Stanford University; Vice-Chairman, Professor C. C. Torrance, United States Naval Postgraduate School, Monterey; Secretary-Treasurer, Professor C. D. Olds, San Jose State College.

Following a recommendation of the Board of Governors, action was taken to make the Sectional Governor a member of the Executive Committee.

By invitation of the Section, Professor G. E. Latta of Stanford University delivered an address at the morning session entitled *Singular Problems for Differential Equations*. Abstract of this address follows:

A singular problem for differential equations is defined as a boundary value problem for a differential system, whose solution depends non-uniformly on a small parameter. Such a problem occurs, for example, when the parameter is the coefficient of the highest order derivatives appearing in the equation. The problem of approximating the solution of the system in terms of the lower order system obtained by neglecting the small parameter is discussed, and asymptotic expansions are obtained in a number of cases, valid for sufficiently small parameter values. Examples are drawn from hydrodynamics, electric circuit theory, and eigenvalue problems, illustrating a heuristic approach; shock waves and boundary layer theory are mentioned as special cases.

The following papers were presented:

1. *Some remarks on the Lindberg case*, by Professor Free Jamison, San Jose State College.

Some extension of the results of the Lindberg trisections (this MONTHLY, vol. 61, pp. 334–336) were given. The method can be used for the approximate division of an angle by any positive integer greater than 3. In particular, the error in approximating an angle of one degree by the use of only 6 lines, 11 arcs, and 2 compass settings is less than two one-hundredths of one second. Remarks included comment on the wide appeal of the problem, the unsophisticated approach, desirability of study of impossible conditions at different school levels, and applicable principles of learning and counseling.

2. *Differentiation of logarithms*, by Professor C. M. Fulton, University of California, Davis.

Two methods are currently used to present the logarithmic functions in calculus. The first one introduces e as a limit. The second approach defines the natural logarithm by means of an integral. Instead of taking the existence of that limit for granted, one can make the equivalent assumption that a logarithm is differentiable. Then well-known algebraic properties of logarithms make it possible to find their derivatives.

3. *Some models and demonstrators for engineering mathematics*, by Professor A. B. Mewborn, United States Naval Postgraduate School, Monterey.

Models developed by the Staff of the Naval Postgraduate School in collaboration with the Special Devices Center of the Office of Naval Research, and built in the shops of the latter activity were shown and explained briefly. Among those discussed were: a 3×linear enlarged Amsler polar planimeter, plastic models illustrating 3-dimensional coordinate systems and vector relationships involving dot and cross products, a wire and plastic model for illustrating the proof and properties of Stokes' theorem, and 3-dimensional models showing the relationship of skew lines in space.

4. *A simple minimum problem*, by Professor F. R. Morris, Fresno State College.

One or more circles are in a plane. They may be tangent but do not overlap. The diameter of each circle is 1 unit. The perimeter is defined as the length of the line composed of tangents and arcs of circles which surround the group of circles. The problem is to arrange the circles in order to have a minimum perimeter and to find its value.

5. *An undergraduate mathematics seminar*, by Professor E. J. Farrell, University of San Francisco, introduced by the Secretary.

A brief report on a seminar offered to upper division mathematics majors, and others with necessary prerequisites, by the University of San Francisco. The objectives were stated, contents of typical seminars shown, and the various types of students enrolling in the course were discussed.

6. *An introduction to determinants*, by Professor C. H. Rawlins, Jr., United States Naval Postgraduate School, Monterey.

Professor Rawlins discussed the traditional approach to the subject of determinants, as found in most of the text and reference books, and suggested that the determinant be introduced in a way that requires no mention of permutations and inversions. He proposed that it be defined as equal to its expansion by minors according to its first row. With this start he derived proofs for the standard elementary theorems. In the proofs, mathematical induction was employed somewhat less than in some previous treatments of the same subject.

7. *An isoperimetric inequality*, by Professor Robert Weinstock, Stanford University, introduced by the Secretary.

Let C be a simple closed plane curve having continuous curvature and bounding a convex domain D. Let A be the area of D, L the length of C, and J the polar moment of inertia of C with respect to its centroid. (For the computation of J it is assumed that a uniform distribution of total mass L is associated with C.) With the use of tangential coordinates it is shown that the inequality $LA \leqq \pi J$—not generally valid for nonconvex domains—holds; equality obtains only if C is a circle.

C. D. OLDS, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The thirty-fourth regular meeting of the Southern California Section of the Mathematical Association of America was held at George Pepperdine College, Los Angeles, on March 13, 1954. Professor A. E. Taylor, Chairman of the Section, presided.

The attendance was one hundred, including the following eighty-one members of the Association:

L. J. Adams, O. W. Albert, Frances C. Ameniya, A. R. Amir-Moez, Norman Anning, T. M. Apostol, Leon Bankoff, J. L. Barnes, Mabel S. Barnes, Lulu Bechtolsheim, May M. Beenken, Clifford Bell, J. S. Bendat, C. J. Biggerstaff, L. T. Black, L. M. Coffin, E. P. Coleman, P. H. Daus, R. P. Dilworth, D. C. Duncan, G. M. Eddington, G. W. Fairchild, A. B. Farnell, C. M. Fast, F. W. Gibson, W. H. Glenn, Jr., B. K. Gold, Jr., J. W. Green, Edison Greer, C. J. A. Halberg, Jr., H. J. Hamilton, V. C. Harris, A. R. Harvey, A. F. Herbst, R. B. Herrera, M. R. Hestenes, R. E. Horton, J. M. Howell, D. G. Humm, H. D. Huskey, D. H. Hyers, C. G. Jaeger, G. R. Kaelin, Rosella Kanarik, P. J. Kelly, L. J. Lander, L. C. Lay, Mary A. Lee, M. M. Lemme, G. F. McEwen, Fred Marer, Helen H. Meek, J. S. Morison, F. R. Morris, R. R. Phelps, A. D. Pierson, W. T. Puckett, Jr., H. R. Pyle, F. C. Reed, E. C. Rex, L. G. Riggs, J. M. Robb, Samuel Skolnik, S. T. Smith I. S. Sokolnikoff, T. H. Southard, Maria A. Steinberg, R. G. Stoneham, J. D. Swift, T. E. Sydnor, P. Y. Tani, A. E. Taylor, Elmer Tolsted, C. B. Tompkins, C. W. Trigg, S. E. Urner, F. A. Valentine, P. A. White, J. V. Whittaker, A. D. Wirshup, W. B. Woolf.

At the business meeting the following officers were elected for the next academic year: Chairman, Professor E. C. Rex, George Pepperdine College; Vice-Chairman, Professor H. F. Bohnenblust, California Institute of Technology; Program Committee: Mr. Fred Marer (Chairman), Los Angeles City College, Mr. L. J. Adams, Santa Monica City College, Professor Elmer Tolsted, Pomona College, Professor A. L. Whiteman, University of Southern California.

The following program was presented:

1. *A discussion of numerical analysis and university curricula*, by Professor Clifford Bell, University of California at Los Angeles, Dr. H. D. Huskey and Dr. C. B. Tompkins, both of the Institute of Numerical Analysis, National Bureau of Standards, University of California at Los Angeles.

Professor Bell described the program in numerical analysis on the Los Angeles campus of the University of California. It was set up as a certificate program under University Extension with the cooperation of the departments of mathematics and engineering and the Institute for Numerical Analysis, National Bureau of Standards. It was designed to give interested persons training in the theory and use of calculating machines, both analogue and digital. A total of 24 semester units are required for the completion of the certificate program. Among the courses required of all candidates are six units in numerical mathematical analysis, six units in the digital calculator and its use, three units in analogue computations, and three units in matrix inversion and decomposition. Outlines of the material presented in these courses were made available to the audience.

Dr. Huskey gave a brief history of the Computation Laboratory at Wayne University. The financial support from industry was outlined and the organizational structure in the University was explained. A list was given of the courses in the mathematics, electrical engineering, and physics departments, which relate to computation. Finally, special lecture series, workshops, and summer programs were briefly described.

Dr. Tompkins discussed the objectives of the program. A modern curriculum in numerical analysis at a University necessarily includes much material relating to modern high-speed automatic digital computers. This imposes a requirement for cooperative effort between various departments of the university: the mathematics department for arithmetic studies, various scientific departments (both physical and social sciences) for the generation of live problems, and engineering departments for the construction and maintenance of machines. The author suggested a reasonable allocation of responsibility for this involved and expensive program.

2. *The theory of dynamic programming*, by Dr. R. E. Bellman, The Rand Corporation, introduced by Professor D. H. Hyers.

The purpose of the talk is to provide an introduction to some of the problems encountered in the study of multi-stage processes and to present some of the techniques that have been devised to treat these problems.

A typical problem is the following: We are given an amount of money, x, to divide into two parts y and $x-y$. From y we obtain a return of $g(y)$, and from $x-y$ a return $h(x-y)$. In so doing, we expend a certain amount of the original quantity and are left with a new quantity, $ay+b(x-y)$, where $0<a, b<1$. This process is now continued. How does one allocate at each stage so as to maximize the total return obtained over a finite or unbounded number of stages?

It is shown that this leads to the functional equation

$$f(x) = \max_{0\leq y\leq x} [g(y) + h(x-y) + f(ay + b(x-y))].$$

3. *Tests for divisibility*, by Professor C. G. Jaeger, Pomona College.

Simple tests for the divisibility of an integer by powers of 2 and powers of 5 are well known. Tests for divisibility by 3 and 9 and 11 are also found in books on algebra and number theory. The test usually given for divisibility by 7 is awkward. In this paper the author gives a simpler test for divisibility by 7, especially when the number tested is large. A similar method is given for divisibility by other primes.

4. *Rational right triangles associated with the arbelos*, by Dr. Leon Bankoff.

The paper treated extensions of properties of the three chains of circles of Pappus inscribed in the Shoemaker's Knife of Archimedes. Lines were drawn connecting the centers of the inscribed circles with the foci of their associated conics. Through the extremities of these lines, pairs of lines were drawn perpendicular to and parallel to the line of centers of the semi-circles forming the arbelos. A necessary and sufficient condition that the right triangles so formed are rational is that the radii of the generating semi-circles are in a rational relationship. Corollaries of this theorem were also discussed.

5. *Some topological invariants in Euclidean spaces*, by Professor E. G. Straus, University of California at Los Angeles, introduced by Professor A. E. Taylor.

Two point sets are *contiguous* if they have a common boundary point but no common interior points; they are *overlapping* if they have an interior point in common. Given a bounded point set S in Euclidean n-dimensional space, then there are $f(S, n)$ nonoverlapping sets which are congruent to S and contiguous to S. For a given S the number $f(S, n)$ is the smallest possible if S is a sphere. Thus $f(S, 2)\geq 6$, $f(S, 3)\geq 12$. (See K. Schütte and B. L. van der Waerden, *Das Problem der 13 Kugeln*, Mathematische Annalen, vol. 125, 1953, pp. 324–325).

Given any closed curve C in n-dimensional Euclidean space and any prescribed triangle T, then there exist three points on C which are the vertices of a triangle similar to T. It is not known whether in the plane we can also prescribe that the interior of T be in the interior of C. (For equilateral triangles, see A. N. Milgram, *Some topologically invariant metric properties*, Proceedings National Academy of Sciences, U.S.A., vol. 29, 1943, pp. 193–195.)

6. *Compact segmental determinants*, by Mr. C. W. Trigg, Los Angeles City College.

From a column of a rectangular array a p, u, n-segment is chosen by taking every uth element, beginning with the element in the pth row, until n have been selected. An array, A, with $a_{1,j}=a_j$, $a_{i,1}=a_1k^{i-1}$, $a_{i,j+1}=ma_{i,j}+ka_{i-1,j+1}$ is a generalized Pascal triangle. The p, u, n-segments from the first n columns of A form a compact equipatterned segmented determinant. The value of this determinant,

$$D = a_1^n(um)^{n(n-1)/2}k^x, \quad \text{where} \quad x = un(n-1)/2 - n + \sum_{j=1}^{n} p_j,$$

is independent of the a_j, $j>1$. For $k=\pm 1$, $|D|$ is independent of the p_j. Among the other special cases, $a_j=m=k=1$ gives the Pascal arithmetical triangle for which $|D|=1$ when $u=1$.

7. *Report on a conference for teachers of collegiate mathematics*, by Professor May M. Beenken, Immaculate Heart College.

Professor Beenken summarized the work of the eight weeks conference on collegiate mathematics held in Boulder, Colorado last summer under the sponsorship of the National Science Foundation. She included suggestions for improving the undergraduate mathematics major and reported on what some colleges are doing to modernize mathematics on the freshman level through the introduction of modern postulational concepts and methods in pre-calculus courses.

P. H. DAUS, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-fifth Summer Meeting, University of Wyoming, Laramie, Wyoming, August 30–31, 1954.

Thirty-eighth Annual Meeting, University of Pittsburgh, Pittsburgh, Pennsylvania, December 30, 1954.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
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METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA
OHIO
OKLAHOMA, Oklahoma City University, October 29, 1954.
PACIFIC NORTHWEST, Reed College, Portland, Oregon, June 18, 1954.
PHILADELPHIA, Princeton University, Princeton, New Jersey, November 27, 1954.
ROCKY MOUNTAIN
SOUTHEASTERN, Tennessee Polytechnic Institute, Cookeville, March 11–12, 1955.
SOUTHERN CALIFORNIA, Santa Monica City College, March 12, 1955.
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Edited by The Institute of Mathematical Statistics; Chairman: T. W. ANDERSON, Columbia University. In press.

Here are the papers of a great American statistician who came to the United States from Austria in 1938 and headed up many important statistics groups for the government and at Columbia University. About 50 articles on statistics and probability are included. The collection starts with the last paper Wald did in Europe and is in German. All the following were published in the United States, many in *Annals of Mathematical Statistics*. The introduction gives perspective to the wide range of areas in which Wald made original contributions and the penetrating quality of his work. A brief biographical sketch is included.

ENGINEERING CYBERNETICS: The Science of Control

By H. S. TSIEN, Daniel and Florence Guggenheim Jet Propulsion Center, California Institute of Technology. In press.

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